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THERMAL SHOCK ANALYSIS OF A BRITTLE MATERIAL

A THESIS

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the Faculty of the Graduate Division

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THERMAL SHOCK ANALYSIS OF A BRITTLE MATERIAL

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SUMMARY

A thermal shock analysis on a thin aluminum oxide disk subjected to a peripheral quench is performed by an analytical approach. That is, the thermal shock analysis is accomplished by solving the governing differential equations for the transient temperatures and thermal stresses in the disk for various values of the surface heat transfer coefficient and the shock temperature. The initial uniform temperature in the disk was held at 1600°F during the analysis. An "exact" analysis, where the variation of the material properties with temperature is taken into account, is given, and a "conventional" analysis, where the material properties are evaluated at a "mean" temperature, is also presented for comparison.

For the "exact" and "conventional" analysis, the temperatures in the disk were found by approximating the non-linear heat conduction equation for polar coordinates by finite-difference expressions and by solving the resulting equation by the Gauss-Siedel iteration procedure. The thermal stresses in the disk were found by approximating the equilibrium and compatibility equations by finite-difference expressions and by solving the resulting set of equations. The solution for the temperatures and thermal stresses was obtained by programming the governing finite-difference equations for the Burroughs 220 digital computer.

The numerical solution of the heat conduction equation proved to be unstable for large values of the heat transfer coefficient and large time increments; however, the instability due to these two factors was overcome by keeping these parameters below their critical values. By

using an alternate form of the differential equation which was valid only at $r = 0$, a means for suppressing the instability which occurred at the center of the disk was found. The numerical solution of the thermal stress equations was found to be quite accurate. The solution was within 2 percent of the correct answers on a test case performed.

A comparison of the results given by the conventional or approximate analysis with the results given by the exact analysis was made. As suggested by Manson, the "mean" temperature which gave the best agreement between the exact and conventional analysis was found to be an average temperature between the initial temperature in the disk and the temperature at which the maximum stress occurred.

GLOSSARY OF ABBREVIATIONS

Material Constants

E	Young's modulus (p.s.i.)
α	linear coefficient of expansion (in/in/°F)
μ	Poisson's ratio
k	thermal conductivity (BTU/sec in °F)
ρ	density (lb/in^3)
c	specific heat capacity (BTU/lb °F)
D	diffusivity (in^2/sec) $\left[= \frac{k}{c\rho}\right]$

Variables

r	radial coordinate in polar coordinate system (inches)
θ	circumferential coordinate in polar coordinate system (radians)
x	X direction in rectangular coordinate system (inches)
y	Y direction in rectangular coordinate system (inches)
t	time (seconds)
$\bar{t}$	non-dimensional time $\left[= (D)(t)/r^2\right]$
T	temperature (deg F)
u	radial displacement (inches)
ϵ	normal strain (in/in)
σ	normal stress (lb/in^2)
σ^*	non-dimensional normal stress $= \frac{\sigma}{E\,\alpha\,(\Delta T)}$

Miscellaneous

Q	heat flux at a surfact (BTU/sec in^2)
$\overline{Q}$	internal heat generation (BTU/sec in^3)
q	heat transfer rate (BTU/sec)
h	heat transfer coefficient (BTU/sec in^2 °F)
M	Modulus $\left[= (\Delta x)^2/(D)(\Delta t)\right]$
N	Nusselt modulus $\left[= (h)(\Delta x)/(k)\right]$
L	radial modulus $\left[= \Delta r/(2)(r)\right]$
β	Biot modulus $\left[= (r)(h)/(k)\right]$

Subscripts

f	fracture
r	radial
t	tangential
o	the center of the disk
a	the outer radius of the disk
m	mean value

Prefix

Δ	a small increment of the indicated variable

CHAPTER I

INTRODUCTION

With the perfection of the turbine and rocket engines, design problems which were caused by thermal stresses became increasingly important. As the efficiencies of these engines were increased, higher temperatures were encountered which caused even greater thermal problems with the materials being used at the time. Many of these thermal problems were solved by using ceramic and ceramal materials to replace the metalic materials, but in using these materials other problems were encountered. Ceramic and ceramal materials are only able to sustain very small strains, and they can not relieve stress concentrations by flowing plastically; therefore, these materials are very susceptible to thermal stress due to this brittle-like behavior.

Since in the majority of these turbine and rocket engines very high thermal gradients are induced upon first starting and stopping the engine, the problems related to "thermal shock" became more important than the ordinary thermal stress problems. Thermal shock is distinguished from ordinary thermal stress by the fact that it is induced by a sudden transient thermal gradient as opposed to a steady state temperature distribution. Actually, there will be no difference in the stress distribution produced by a thermal shock and a steady state temperature distribution if the thermal gradients are the same; however, much higher thermal gradients can be obtained during a thermal shock than can be obtained under steady state conditions.

Considerations could also be made of repetitive applications of a thermal shock which encompasses the concept of thermal fatigue. However, in this study the term "thermal shock" will denote the application of a thermal gradient of one cycle only since we are primarily interested in a brittle type material. All failures considered will be caused by a thermal shock of one cycle only.

Early investigators of thermal shock (see References 19, 21, 22) usually analyzed a simple body, a flat plate or disk, under the action of a thermal shock at its surface. By using the analytical results obtained for the transient stress distribution, several parameters are found which describe the failure of the body under the thermal shock. The results obtained usually give the maximum temperature difference the body can be suddenly subjected to without failure, for various values of the thermal shock parameters. All of these studies are based upon the assumption that the mechanical and physical properties of the body remain constant for the duration of the thermal shock; however, this assumption can introduce considerable error in the analysis since the properties of ceramic and ceramal materials vary drastically with temperature. Attempts have been made to compensate for these errors by evaluating the material properties at various intermediate temperatures; however, no comparisons have been made that substantiate the accuracy obtained by using some intermediate temperature to evaluate the material properties. For this reason, the current investigation was undertaken.

The object of this study was to obtain an analytical solution, by taking into account the variation of the mechanical and physical properties with temperature, for one of the typical bodies usually analyzed in a thermal shock analysis, and to compare these results with those obtained by evaluating the properties at "average" or "mean" temperatures.

The body selected for analysis was a thin disk (radius = 1 inch) subjected to a thermal shock on the outer periphery and insulated over the flat faces. This is the type of body used by Manson[20] to run experimental tests. The material chosen as representative of most ceramics was aluminum oxide (Al_2O_3). The variation with temperature of modulus of elasticity, the coefficient of expansion, Poisson's ratio, thermal conductivity, thermal diffusivity, and tensile fracture stress was found in the literature[22] and these properties fitted to curves through the use of a digital computer program for fitting non-linear curves to data[1]. A numerical approach to solving the thermal shock problem was decided upon since the differential equations were non-linear due to the dependency of the material properties on the temperature.

Chapter II gives some definitions, notation and fundamental concepts of numerical analysis which will be needed in the discussion of the solution to the temperature and thermal stress equations in Chapters III and IV. Chapter V covers the general methods used in thermal shock analysis of brittle materials as well as the method of approach used in this study. In Chapter VI, the conclusions and recommendations resulting from this study are presented.

CHAPTER II

NUMERICAL ANALYSIS

In order to formulate and solve the temperature and thermal stress problems by using numerical analysis, the notation to be used as well as some definitions must be set forth. Very briefly, the procedure will be to replace the differential equations and the boundary and initial conditions with finite-difference expressions for the veri-ous partial and ordinary derivatives.

Basic Definitions

It is possible to establish the finite-difference relations by several methods. Two of these methods are the use of Taylor's series and the use of linear approximations to the function over small incre-ments. These relations will be established for simple derivatives and then generalized for partial derivatives.

Referring to Figure 1, the Taylor's series expansion around the point x=i is

$$y = f(x) = f(i) + (x-i)f'(i) + \frac{(x-i)^2}{2!} f''(i) + \frac{(x-i)^3}{3!} f'''(i) + \dots$$

This expression can be used to obtain the finite difference approxi-mation to the first derivative of y=f(x) at a point such as "i". This is accomplished by evaluating the series at the points on either side of "i" and subtracting them. For instance:

$$f(i+1) = f(i) + b\, f'(i) + \frac{b^2}{2!} f''(i) + \frac{b^3}{3!} f'''(i) + \dots \quad (1)$$

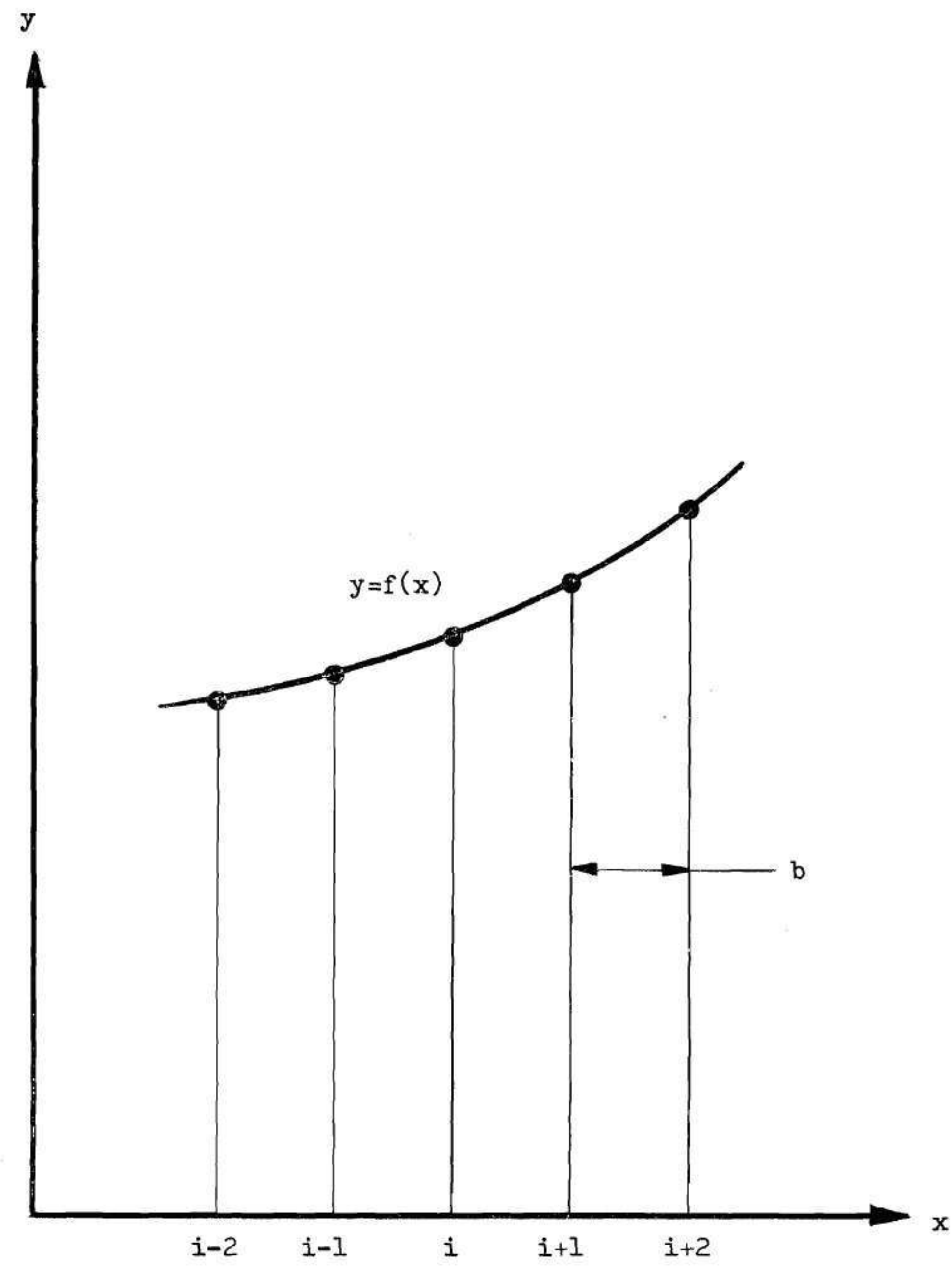

Figure 1. Finite-Difference Grid for Ordinary Derivatives.

and

$$f(i-1) = f(i) - b\ f'(i) + \frac{b^2}{2!} f''(i) - \frac{b^3}{3!} f'''(i) + \ldots \tag{2}$$

when expressions (1) and (2) are subtracted, they yield

$$f(i+1) - f(i-1) = 2b\ f'(i) + \frac{2b^3}{3!} f'''\ (i) + \ldots \tag{3}$$

or the expression for the first derivative at the point "i" becomes

$$f'(i) = \frac{f(i+1) - f(i-1)}{2b} - \frac{b^2}{3!} f'''\ (i) + \ldots \tag{4}$$

In a similar manner, the second derivative of $y=f(x)$ could be obtained by adding expressions (1) and (2).

$$f(i+1) + f(i-1) - 2\ f(i) = b^2\ f''(i) + \frac{2b^4}{4!} f''''(i) + \ldots$$

or, the expression for the second derivative at the point "i" becomes

$$f''(i) = \frac{f(i+1) - 2\ f(i) + f(i-1)}{b^2} + e(b^2) \tag{5}$$

where

$e(b^2) = \frac{2b^2}{4!} f''''(i) + \ldots =$ the "truncation error" involved which is of the order of b^2.

This truncation error is related to the convergence of the numerical solution to the actual or analytical solution, which will be discussed later.

The second method of obtaining the finite-difference expressions of the simple derivatives could be thought of as merely fitting a linear curve between adjacent points on the y=f(x) curve. Referring to Figure 1, the first derivative or slope at the point "i" could be approximated by

$$f'(1) = \frac{f(i+1) - f(i)}{b} \tag{6}$$

or by

$$f'(i) = \frac{f(i) - f(i-1)}{b} \tag{7}$$

Another valid representation would be

$$f'(i) = \frac{f(i+1) - f(i-1)}{2b} \tag{8}$$

In the above equations, the slope of the curve was merely approximated at the point ($i=\pm b/2$) in equations (6) and (7) and at the point (i) in equation (8).

We can approximate the second derivatives by using equation (6) and (7) as follows:

$$f''(i) = \frac{f'(i+b/2) - f'(i-b/2)}{b}$$

$$f''(i) = \frac{[f(i+1) - f(i)] - [f(i-1) - f(i)]}{b^2}$$

Finally, the approximation for the second derivative becomes

$$f''(i) = \frac{f(i+1) - 2\,f(i) + f(i-1)}{b^2} \tag{9}$$

The only disadvantage of this second method is that it gives no estimate of the truncation error. However, this error is readily determinable from the Taylor's series analysis. For expressions (6) and (7), the truncation error is of the order of "b". For expressions (8) and (9) the error is of the order of "b^2". Thus, for small values of "b", expression (8) would give better approximations for the first derivative than either expressions (6) or (7).

In order to extend our finite-difference approximations to partial derivatives, we must set up a region of the XY plane which is subdivided into rectangles by a grid or mesh of lines which are parallel with the X and Y axes. This region is shown in Figure 2.

Without much change, we can use our relations for ordinary derivatives to establish the relations for partial derivatives. Referring to Figure 2 and the relations (8) and (9), we have

$$\frac{\partial}{\partial x} f(i,j) = \frac{f(i+1,j) - f(i-1.j)}{\Delta x} \tag{10}$$

$$\frac{\partial}{\partial y} f(i,j) = \frac{f(i,j+1) - f(i,j-1)}{\Delta y} \tag{11}$$

$$\frac{\partial^2}{\partial x^2} f(i,j) = \frac{f(i+1,j) - 2\,f(i,j) + f(i-1,j)}{(\Delta x)^2} \tag{12}$$

$$\frac{\partial^2}{\partial y^2} f(i,j) = \frac{f(i,j+1) - 2\,f(i,j) + f(i,j-1)}{(\Delta y)^2} \tag{13}$$

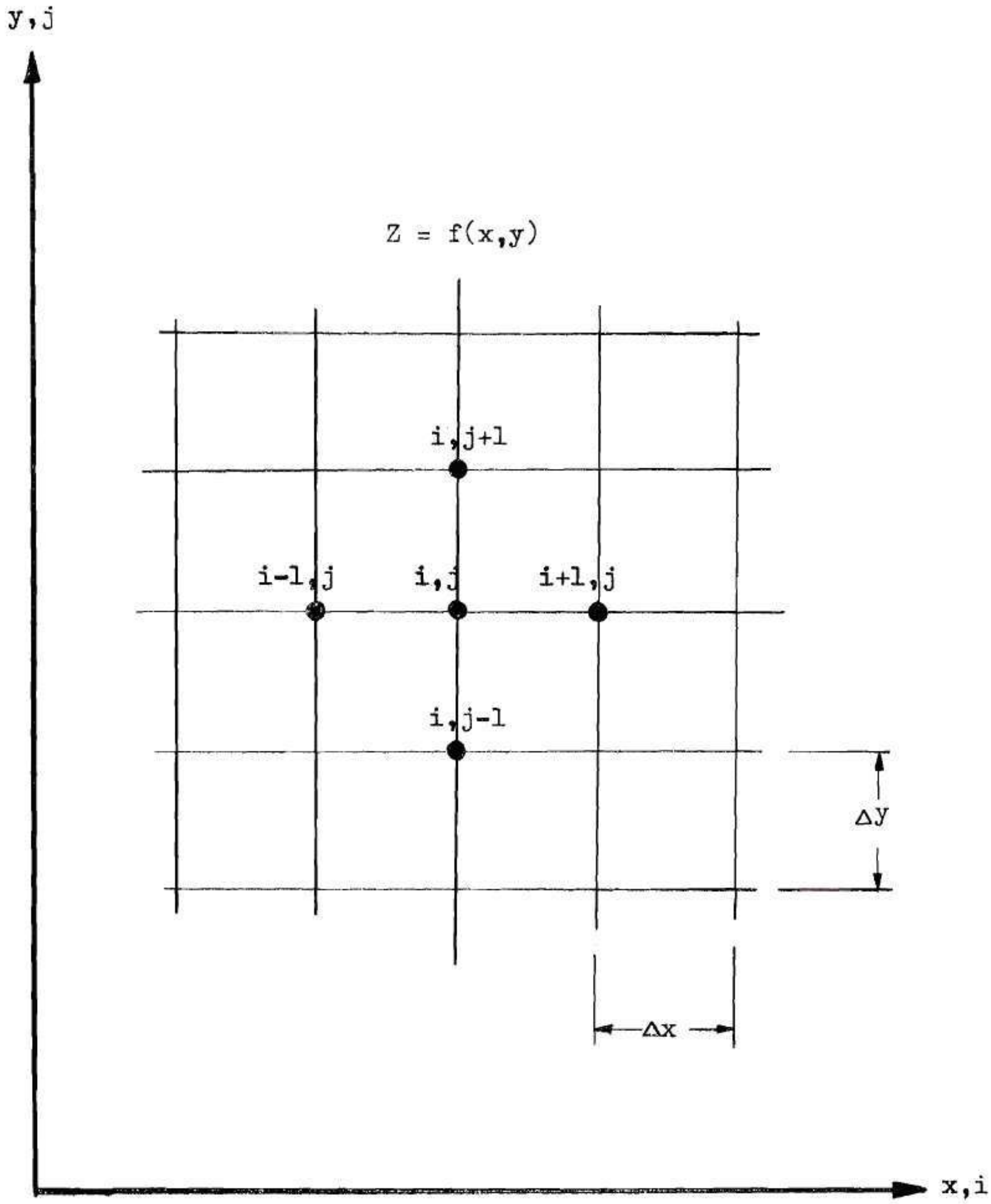

Figure 2. Finite-Difference Grid for Partial Derivatives.

By combining the above relations, the Laplacian and Biharmonic operators can be obtained. Provided that $\Delta x = \Delta y = b$, the Laplacian operator would be

$$\nabla^2 f(i,j) = \frac{f(i+1,j) + f(i-1,j) + f(i,j+1) + f(i,j-1) - 4f(i,j)}{b^2} \tag{14}$$

Thus, the relations needed to formulate the numerical solution of the heat conduction equation and the thermal stress equation have been established. However, some additional considerations are needed in order to obtain the solution of the heat conduction equation.

Convergence and Stability

For partial differential equations of the parabolic type[3] (governing heat conduction and diffusion), the relations dealing with the stability and convergence of numerical solutions are particularly important. For some of the finite-difference approximations, the numerical solution will only be stable and converge for certain ratios of the grid size and certain values of the parameters. The prediction of the stability of a certain numerical problem can become very involved, and in the majority of the problems, no exact estimate can be found. Some definitions and general remarks will be made here so that the stability and convergence of the thesis problem can be discussed. A rather extensive discussion of convergence and stability can be found in References (2) and (3).

In order to discuss the accuracy with which a numerical solution to a differential equation approximates the actual solution, the terms "truncation error" and "numerical error" must be defined. "Truncation error" can be defined as the discrepancy between the exact solution of the partial differential equation and the exact solution of the finite-difference equation. Provided the numerical solution does converge to an answer, this answer will be closer to the exact answer when there are smaller truncation errors involved in the finite-difference expressions. "Numerical error" or "round-off error" is defined as the discrepancy between the exact solution of the finite-difference equation and the actual numerical answers obtained from this equation. This is directly related to the "stability" of the numerical solution. That is, instability results when a numerical error is amplified as the solution proceeds.

It is most convenient to discuss the convergence and stability of the numerical solution of the heat conduction equation in terms of the equation with only one space dimension[3]. In the majority of cases, any convergence or stability criterion that is established for this case can be applied to the more general case. Thus, the equation which will be examined is[6]

$$D \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{15}$$

Using the results of the last section, we can express equation (15) in finite-difference form. There are many expressions that could be used to approximate the time or the space derivatives. Three expressions which would be suitable approximations for the time derivative are given by equations (6), (7) and (8). Of course, equation (9) would be a suitable approximation for the space derivative.

By referring to Figure 3 and using expressions similar to equations (6) and (9), we can express equation (15) in the simplest finite-

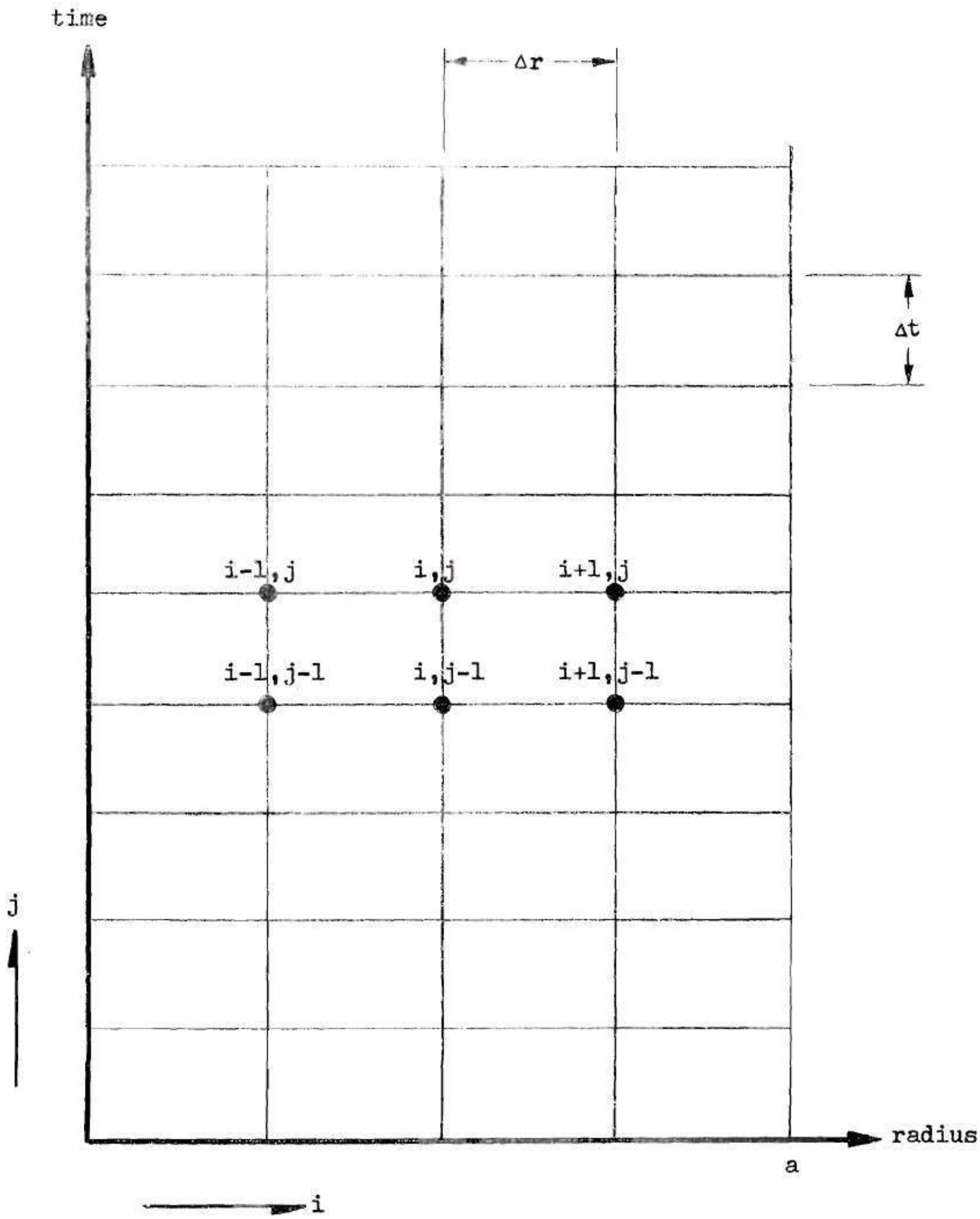

Figure 3. Finite-Difference Grid for the Temperature.

difference form:

$$\frac{T(i+1,j) - 2\,T(i,j) + T(i-1,j)}{(\Delta x)^2} = \frac{1}{D}\left[\frac{T(i,j+1) - T(i,j)}{t}\right] \tag{16}$$

Solving for T(i,j+1) gives

$$T(i,j+1) = \frac{1}{M}\left[T(i+1,j) + T(i-1,j)\right] + (1 - 2/M)\left[T(i,j)\right] \tag{17}$$

where

$$M = \frac{(\Delta x)^2}{D(\Delta t)} \tag{18}$$

Thus, it is possible by using equation (17) to solve for the temperature at any node point (i,j+1) in terms of known temperatures at surrounding node points. Equation (17) is frequently termed the "explicit" equation.

A stability criterion for equation (17) is established formally in references (2), (3) or (6), and it is found that for stability the following must hold

$$M \geq 2 \tag{19}$$

In most cases, this criterion will restrict one to relatively small Δt's which would require many time increments if a solution for a problem of long duration were required. This is the main disadvantage in using the so called "explicit" equation, equation (17), for the solution of the heat conduction equation.

The disadvantage of being restricted to small time increments when using equations (17) led many authors to propose other finite-difference approximations to equation (15). Liebmann[8] and Crank and Nicolson[7] have proposed finite-difference approximations to equation (15) which are stable for all values of the modulus, "M". Both of these solutions are of the "implicit" type because when solving for T(i,j+1) all the required temperatures are not known. This requires an iterative type solution. The particular finite-difference approximations chosen by Crank and Nicolson have truncation errors of the order of $(\Delta x)^2$ and $(\Delta t)^2$, but the approximations chosen by Liebmann have truncation errors only of the order of $(\Delta x)^2$ and (Δt). This fact can be seen by looking at the proposed finite-difference approximations for equation (15) and comparing the terms with relations (6), (7), (8) and (9). The equation proposed by Liebmann is:

$$\frac{T(i+1,j) - 2\,T(i,j) + T(i-1,j)}{(\Delta x)^2} = \frac{1}{D}\left[\frac{T(i,j) - T(i,j-1)}{\Delta t}\right] \tag{20}$$

The equation proposed by Crank and Nicolson is :

$$\frac{T(i+1,j+\frac{1}{2}) - 2\,T(i,j+\frac{1}{2}) + T(i-1,j+\frac{1}{2})}{(\Delta x)^2} = \frac{1}{D}\left[\frac{T(i,j) - T(i,j-1)}{2\,\Delta t}\right] \tag{21}$$

Thus, the finite-difference approximations proposed by Crank and Nicolson can be expected to converge to answers nearer the true solution due to the smaller truncation error in the time derivative.

In heat conduction problems where the heat flux is specified on the boundary, another stability criterion, in addition to one like equation (19), must be satisfied. Schneider[10] gives the following criterion for stability at the boundary when the "explicit" equation is used

$$M \geq 2N + 2 \tag{22}$$

where

$$N = \frac{h \ \Delta x}{k} \tag{23}$$

Thus, if the heat transfer coefficient, h, gets too large relative to the other parameters, instability may result.

For the implicit or stable equations like those proposed by Crank and Nicolson, a stability criterion such as that given by equation (22) should not have to be satisfied. Tests were run on the stability of equation (21) for very high values of "h" by the writer. The solutions were found to oscillate; however, convergence did occur. Thus, the conclusions made were that the solution to the heat conduction equation in rectangular coordinates by the method of Crank and Nicolson would be stable for large values of the heat transfer coefficient.

For the heat flow problem in polar coordinates, no stability criterion has been established to this writers knowledge. The equation governing heat conduction in the radial direction only, equation (27), has a lower order term, namely $(1/r)(\partial T/\partial r)$. Richtmyer[3] states that for an equation of the following type

$$A \frac{\partial^2 T}{\partial r^2} + B \frac{\partial T}{\partial r} + C\ T = \frac{\partial T}{\partial t} \tag{24}$$

where

A, B, and C are constants and $A > 0$

the stability is practically unaffected by the lower order terms. Of course, in the heat conduction equation for polar coordinates, equation (27), the "B" term is not a constant but it is a variable ($B = 1/r$); therefore, no such statement could be made about the stability of this equation. As it turns out, the stability of the solution to equation (27) is affected by this lower order term. This will be discussed in the next chapter.

CHAPTER III

SOLUTION OF THE TEMPERATURE PROBLEM

General Formulation

The Fourier heat conduction equation for polar coordinates, which is derived in Appendix A, is as follows:

$$\frac{\partial}{\partial r}\left(k \frac{\partial T}{\partial r}\right) + \frac{k}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right) + \overline{Q} = \frac{\partial}{\partial t}(\rho c T) \qquad (25)$$

If the temperature distribution is independent of the tangential direction and there is no internal heat generation, the equation is still non-linear but it becomes much simpler. This is the equation which governs the temperature distribution in a thin disk insulated at its faces and subjected to a uniform thermal shock on its periphery:

$$\frac{\partial}{\partial r}\left(k \frac{\partial T}{\partial r}\right) + \frac{k}{r}\frac{\partial T}{\partial r} = \frac{\partial}{\partial t}(\rho c T) \qquad (26)$$

If the material constants were treated as being independent of the temperature, equation (26) would become a rather simple linear differential equation, as follows:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{D}\frac{\partial T}{\partial t} \qquad (27)$$

The initial and boundary conditions for the thermal shock problem become

$$T(r,0) = T_z \qquad (28)$$

$$\frac{\partial}{\partial r} T(0,t) = 0 \qquad (29)$$

$$\frac{\partial}{\partial r} T(a,t) = \frac{Q}{k} \qquad (30)$$

Equation (28) merely states that, at the instant the thermal shock begins at the periphery of the disk, the temperature is equal to constant value of "T_z". Equation (29) states that the temperature of the disk is symmetrical about the centerline since the temperature gradient is zero there. Equation (30) states that the temperature gradient at the periphery of the disk is proportional to the heat flux at that point.

In general, it is very difficult to find a solution to equation (25), even with the tangential derivatives dropped, due to its non-linearity. There are various ways to linearize the equation by making a substitution for the dependent or the independent variable[12]. Even with the problem linearized, a general analytical solution would be difficult if not impossible to obtain. However, by using a numerical approach to the problem, the non-linearity can be easily handled. This is accomplished by treating the non-linear terms as known constants, and changing them from time to time as the iterative solution proceeds. How this is accomplished will be shown in the next section.

Numerical Formulation

Since the effect of the $(1/r)(\partial T/\partial r)$ term on the stability of the numerical solution of the heat conduction equation was unpredictable, the finite-difference approximation which gave the most stable solution for any "M" and "N" and had the smallest truncation errors was desired. Therefore, an approach similar to equation (21), the method of Crank and Nicolson, was decided upon. That is, central differences are used to approximate all the derivatives, and the finite-difference relations are written for the node point r=i and t=j-½. These same relations are also used for the initial and boundary conditions. The details of the derivation are given in Appendix B. By using equation (4B) from Appendix B, the temperature at the general node (i,j) becomes:

$$T(i,j) = \left[1/2 + 2M(i,j)\right] \left[\left[2M(i,j) - 2\right]\left[T(i,j-1)\right] + \left[1 + L(i)\right]\left[T(i+1,j) + T(i+1,j-1)\right] + \left[1-L(i)\right]\left[T(i-1,j) + T(i-1,j-1)\right]\right] \tag{31}$$

Equation (31) will be used to solve for the temperature at each node in the disk except the center and peripheral nodes. As explained in a later paragraph, the temperature at the center node is found by using a modified differential equation, equation (32), or its finite-difference approximation, equation (6B). The temperature at the peripheral node is found by using the boundary conditions. This derivation is also given in Appendix B.

Since all the temperature terms on the right hand side of equation (31) are not known when the temperature at the general point (i,j) is being computed, the equation is implicit, and an iterative solution is required. That is, equation (31) is solved many times for the temperature at each node as the solution proceeds. Each time it is solved, the temperatures come closer to the correct values.

The M(i,j) terms are the non-linear terms in the equation since the coefficient D is a function of the temperature. As previously mentioned, they are treated as constants during each iteration. That is, they are evaluated at the beginning of an iteration by assuming they are functions of the temperature at an earlier iteration. The details of the determination of the temperature at which to evaluate the non-linear terms previous to an iteration can be found in Appendix B.

A computer program to solve equation (31) subject to the boundary and initial conditions was written in "ALGOL" language for use on the Burroughs 220 digital computer at the Georgia Institute of Technology. As a check on the stability of this equation, preliminary tests were run with constant properties for various Δt's, heat transfer coefficients, and length moduli, L(i). It was found that the solution was not stable for all combinations of these stability parameters as was the case for the finite-difference equation (21) for rectangular coordinates.

The length modulus, L(i), was found to have a profound effect on the stability of the solution. That is, a limit on the size of L(i) at the center node was found to be a strong criterion for stability. If L(i) was made too large (the center node taken very near $r = 0$), the solution always diverged regardless of any adjustments made in Δt or the heat transfer coefficient. This type of instability was characterized by the solution starting to diverge at the center nodes and proceeding to the outer nodes as the iterations increased. The only way to stop this divergence due to the length modulus was to keep L(i) small. This required keeping the inner node at a value of the radius of about 0.01 inches. Since this seemed to be a rather artificial method of insuring stability at the center node, a better method was desired. By using L'Hospital's rule, the limit as $r \to 0$ of the $1/r(\partial T/\partial r)$ term was found to be $\partial^2 T/\partial r^2$. Therefore, it was possible to use this result to arrive at a new differential equation which would be valid only at the center ($r=0$) of the disk. This equation is as follows:

$$\frac{\partial^2 T}{\partial r^2} = \frac{1}{2D} \frac{\partial T}{\partial t} \tag{32}$$

Through the use of equation (32) at the center node, the instability caused by the L(i) term was completely suppressed.

It was found that for large values of Δt, the solution became unstable similar to the behavior of the explicit equation (17) when the stability criterion (19) was violated. This type of instability was characterized by the divergence beginning at all nodes simultaneously. To remedy this behavior, the time increment in the computer program was reduced to the point where a stable solution was obtained when the heat transfer coefficient, h, had a value which was most detrimental to stability.

For high values of the heat transfer coefficient, the solution became unstable in a manner similar to the behavior of the "explicit" equation (17) when the second stability criterion, equation (22), is violated. This type of instability was characterized by the solution first diverging at the peripheral node and then spreading into the interior of the disk. Since the thermal shock problem had to be solved for several different heat transfer rates, several values of the heat transfer coefficient, "h", had to be used regardless of the instability of the solution for high surface heat flux. The only method to circumvent this problem was to choose Δt small enough so that no instability due to a high "h" was induced. That is, the stability criterion analogous to equation (22) for the explicit problem was satisfied by reducing the value of Δt.

CHAPTER IV

SOLUTION OF THE THERMAL STRESS PROBLEM

In order to solve for the thermal stresses in a thin disk with a radial temperature distribution, only the basic isotropic elasticity relations need be used.

To obtain the equilibrium equation, the forces are summed on an element such as that shown in Figure 4. Summing forces in the radial direction yields

$$(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr)(r+dr)d\theta dz - \sigma_r(rd\theta dz) - \sigma_t(drd\theta dz) = 0$$

If σ_t is independent of θ, a summation of forces in the tangential direction yields

$$\sigma_t \, drdz(\cos \frac{d\theta}{2}) - \sigma_t \, drdz(\cos \frac{d\theta}{2}) = 0 \qquad (33)$$

Since equation (33) is identically satisfied, the only equilibrium equation is

$$\frac{d}{dr}(\sigma_r) + \frac{\sigma_r - \sigma_t}{r} = 0 \qquad (34)$$

Since the equilibrium equation is in terms of the stresses, a compatibility relation must be established. The strain-deformation relations for a plane stress and extensional deformation problem (the disk is assumed not to bend) are[14]

$$\epsilon_r = \frac{du}{dr} \qquad (35)$$

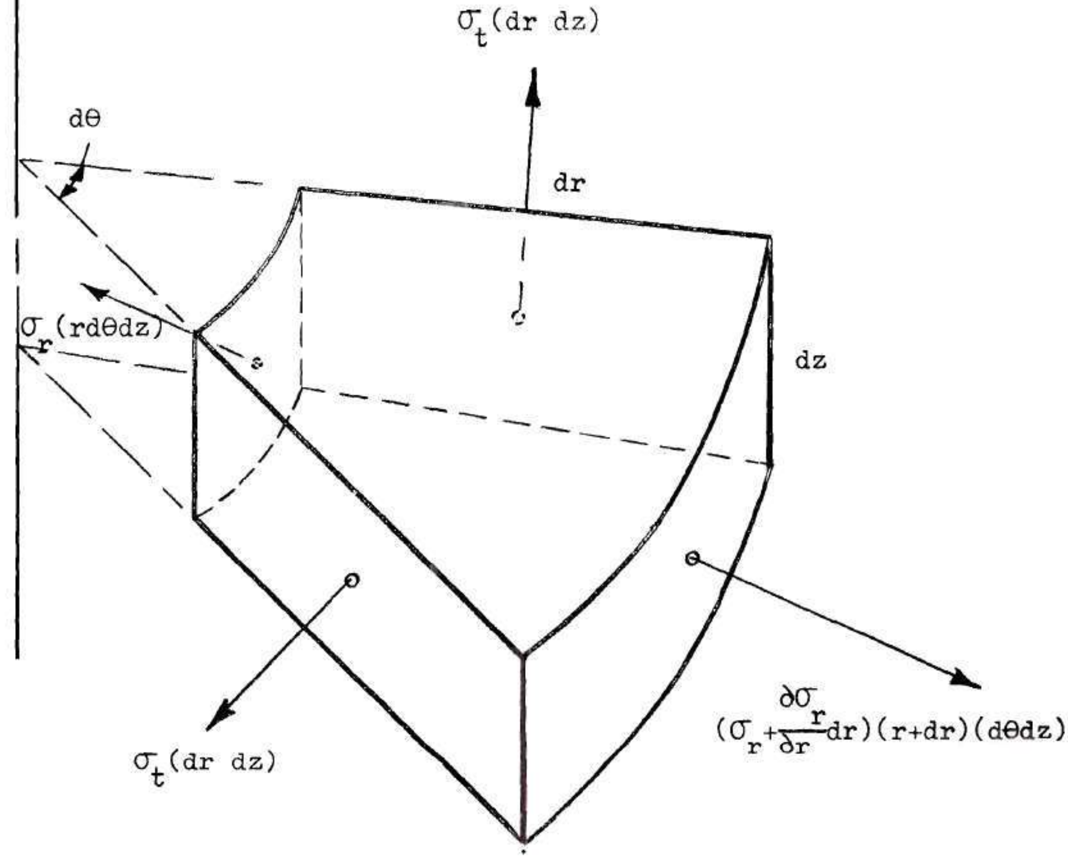

Figure 4. Equilibrium of an Elementary Particle in Polar Coordinates.

and

$$\epsilon_t = \frac{u}{r} \tag{36}$$

Since the material being considered is very brittle, only linear stress-strain relations are considered. Therefore, the stress-strain relations for this isotropic material are

$$\epsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_t) + \int_{T_o}^{T} \alpha(T)\, dt$$

and

$$\epsilon_t = \frac{1}{E}(\sigma_t - \mu\sigma_r) + \int_{T_o}^{T} \alpha(T)\, dT$$

However, if an average coefficient of expansion, $\bar{\alpha}$, is defined as follows:

$$\bar{\alpha} = \frac{1}{T_o - T}\int_{T_o}^{T} \alpha(T)\, dT$$

then the above equations can be expressed as:

$$\epsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_t) + \bar{\alpha}\Delta T \tag{37}$$

$$\epsilon_t = \frac{1}{E}(\sigma_t - \mu\sigma_r) + \bar{\alpha}\Delta T \tag{38}$$

By equating equation (35) and (37) and equation (36) to (38), the stress-deformation relations are obtained:

$$\frac{du}{dr} = \frac{\sigma_r - \mu\sigma_t}{E} + \bar{\alpha}\,\Delta T \tag{39}$$

and

$$\frac{u}{r} = \frac{\sigma_t - \mu\sigma_r}{E} + \bar{\alpha}\,\Delta T \tag{40}$$

Eliminating $\bar{\alpha}\,\Delta T$ from equations (39) and (40), the following is obtained

$$\frac{du}{dr} - \frac{u}{r} = \frac{1+\mu}{E}\,(\sigma_r - \sigma_t) \tag{41}$$

If both sides of equation (41) are divided by "r", the left side will become a total differential.

$$\frac{r\,du - u\,dr}{r^2 dr} = \frac{1+\mu}{E\,r}\,(\sigma_r - \sigma_t) \tag{42}$$

or

$$\frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1+\mu}{E\,r}\,(\sigma_r - \sigma_t) \tag{43}$$

Substituting from equation (40) into equation (43) yields

$$\frac{d}{dr}(\sigma_t/E) - \frac{d}{dr}(\mu\sigma_r/E) + \frac{d}{dr}(\bar{\alpha}\,\Delta T) = \frac{1+\mu}{E}(\sigma_r - \sigma_t) \tag{44}$$

This is the compatibility equation.

The boundary conditions for the problem are

$$\sigma_r(a) = 0 \tag{45}$$

and

$$\sigma_r(o) = \sigma_t(o) \tag{46}$$

The two unknowns, σ_r and σ_t, can be found by solving the equilibrium equation (34) simultaneously with the compatibility equation (44) and the boundary conditions.

Since E, $\bar{\alpha}$ and μ are functions of the temperature and the temperature distribution was solved by using numerical analysis, a numerical analysis approach was decided upon for the simultaneous solution of the equilibrium and compatibility equations. The formulation of these two equations in finite-difference form is given in Appendix C.

CHAPTER V

THERMAL SHOCK

In this chapter, the basic method of analysis of thermal shock problems, where the material properties are independent of the temperature, is presented. Then, the exact type of analysis used on the present problem, where the material properties are considered as functions of the temperature, is explained. Finally, a discussion is given concerning the temperature at which any temperature-dependent properties should be evaluated in making an exact analysis.

Basic Method of Solution

The resistance of a brittle material to fracture or failure by a thermal shock can be described in a "qualitative" or a "quantitative" manner. Early investigations[31,32] into the behavior of materials under thermal shock were predominately of the qualitative type. That is, several materials of a given size and shape were tested under identical conditions to determine which had the best resistance to failure by thermal shock. However, the results of tests such as these showed that the resistance of a material to fracture depended on the type of test conducted as well as the material being used. One material might have a superior resistance to thermal shock fracture in one set of tests but be inferior in a set of tests conducted under different heating conditions. These results pointed out the need for a "quantitative" evaluation of thermal-shock resistance.

From a quantitative standpoint, "thermal-shock resistance" can be defined as the lowest temperature difference a given body with a given heat transfer coefficient, h, at its surface can withstand without fracture or failure. A measure of the thermal-shock resistance can be obtained once the pertinent thermal-shock parameters are known. These parameters depend on such things as the shape of the body under consideration, the magnitude of the temperature gradients induced in the body, the physical and material properties of the body, and the fracture criterion used. Actually, these parameters could be combined to yield two parameters; therefore, the thermal-shock resistance could be expressed as

$$(T_z - T_{amb}) = f(MF \cdot SF) \tag{47}$$

where

MF = a function of the material and the fracture criterion.

SF = a function of the shape of the body and the temperature distribution.

These parameters, MF and SF, are not independent of each other because the selection of a value of the shape factor, SF, may dictate the form of the material factor, MF, to be used. The reasons behind this behavior will be pointed out later.

The first step in determining these thermal-shock parameters is to solve for the transient temperature distribution and the resulting thermal stress distribution by an analytical or numerical method. With the transient stress distribution known, a plot of the thermal stress at any point in the body (usually taken as the maximum stress) versus time for various values of the heat flux at the boundary (caused by different values of "h") could be made. If these three quantities are expressed in non-dimensional form, a graph similar to the one shown in Figure 5 will be obtained[20]. From this graph, the maximum non-dimensional stress in the body, σ_m^*, can be found for each value of the Biot modulus, β. Then, a plot of σ_m^* versus β can be made (see Figure 6), and a mathematical expression fitted to this curve. From this expression, an equation similar to the following would be obtained:

$$\sigma_m^* = \frac{\sigma_m}{E \alpha (T_z - T_{amb})} = F(R\ h/k) \tag{48}$$

where

R = a characteristic length of the body

In order to proceed further, a choice of a failure criterion must be made. Without going into the details of the selection of a good fracture criterion (see Reference 20), it will suffice to say that for most thermal shock problems where the shock tends to cool the body, the fracture criterion would be that failure occurs when the normal tensile stress in the body reaches the tensile fracture stress of the material. If this fracture criterion is used, an equation can be obtained which gives the temperature difference which will cause a tensile failure to occur in the body. Setting $\sigma_m = \sigma_f$, and using equation (48), the following is obtained

$$\sigma_f^* = \frac{\sigma_f}{E \alpha (T_z - T_{amb})} = F(R\ h/k) \tag{49}$$

or, the thermal-shock resistance is given by

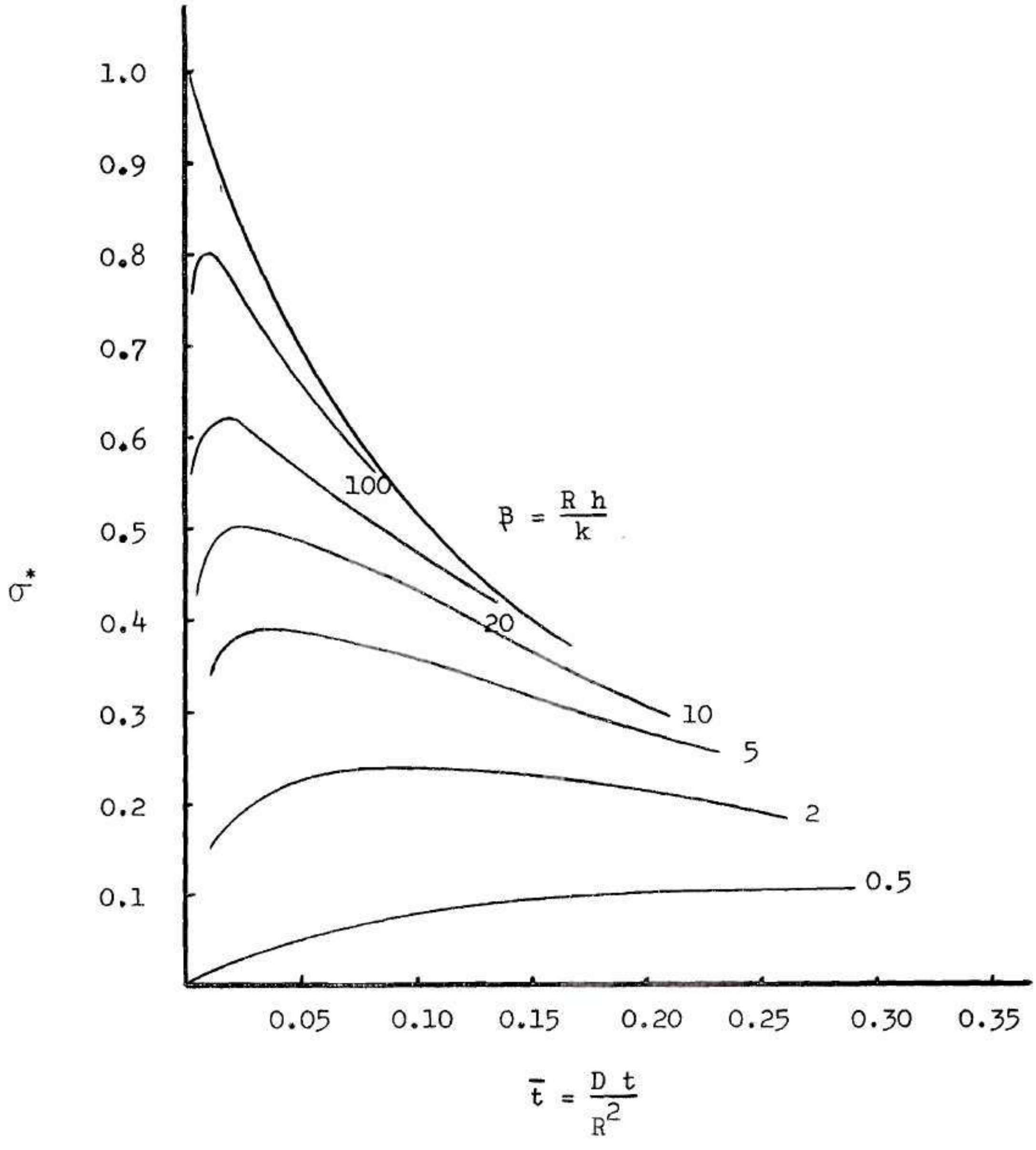

Figure 5. Dimensionless Stress versus Dimensionless Time.

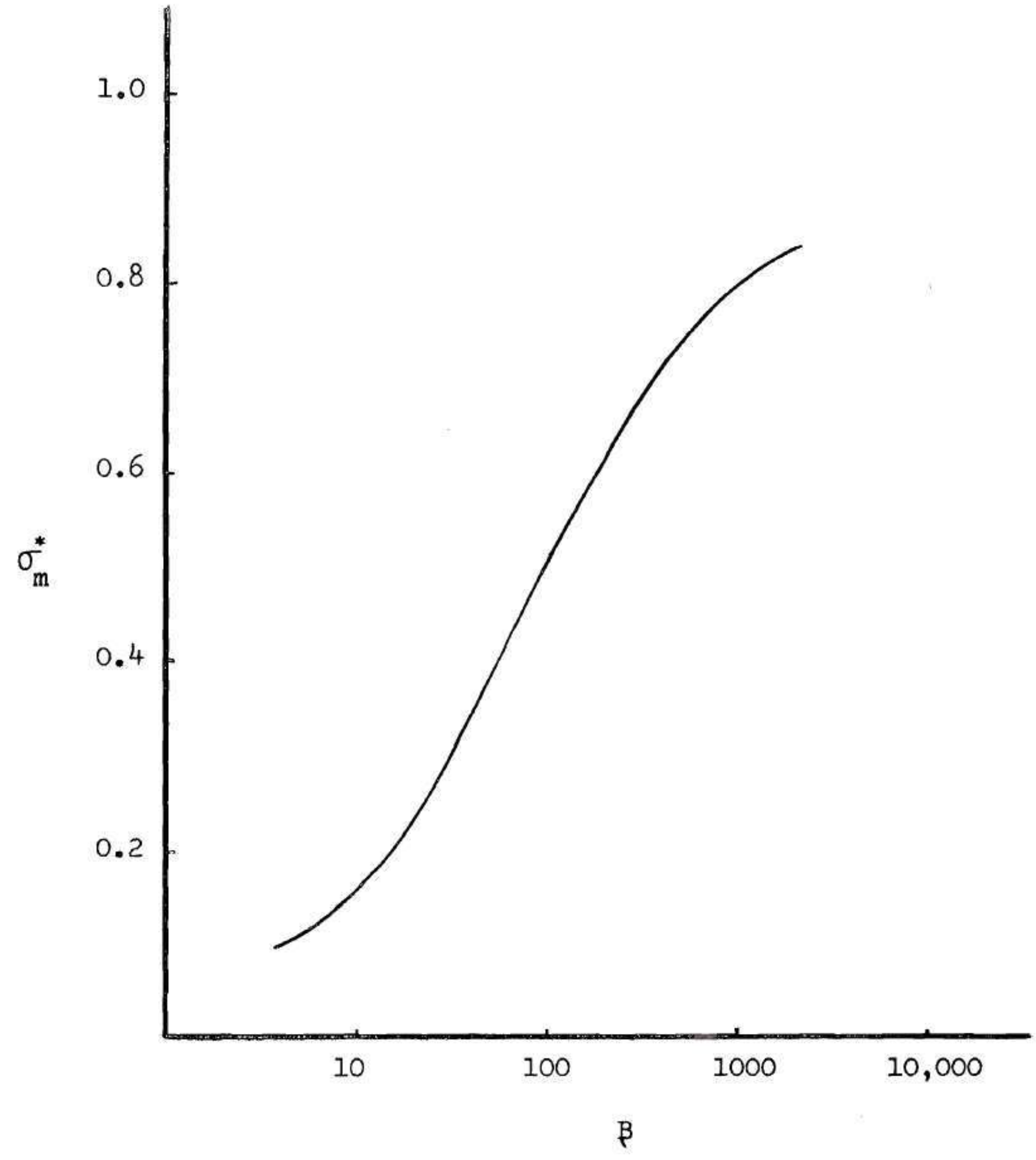

Figure 6. Maximum Dimensionless Stress versus Biot Modulus.

$$T_z - T_{amb} = \frac{\sigma_f}{(E\ \alpha)\ \ F(R\ h/k)} \tag{50}$$

Referring to equation (50) and (47), the two thermal-shock parameters in this case could be

$$MF = \frac{\sigma_f}{E\ \alpha}\ K(k) \tag{51}$$

$$SF = H(Rh) \tag{52}$$

For the case being illustrated, Poisson's ratio happened not to appear in the material factor, MF; however, in general this is not the case. There is also the possibility that the diffusivity of the material or a factor relating the material's susceptibility to flaws (Weibull's theory) would appear in this material factor (see Reference 22). As previously mentioned, the form of the material factor may be dependent upon the shape factor, SF, of the body. The reason for this is that for a large or massive body (R very large) or for a high thermal shock (h very large), the value of the thermal conductivity, k, has no effect on the thermal stress distribution because the maximum stress will occur initially. In a case like this, the material factor would be independent of "k" and would be given by

$$MF = \frac{\sigma_f}{E\ \alpha} \tag{53}$$

Since the analysis mentioned above is based on the material properties being independent of the temperature, a choice of material would make the thermal shock resistance a function of the shape modulus, SF. This would enable a plot to be made of the thermal-shock resistance, $(T_z - T_{amb})$, versus the shape factor. A similar plot has been made by Manson[16] for two materials, Al_2O_3 and BeO; however, Manson plots T_z, $(T_{amb} = 0)$, versus "a h" (where a = R). This plot is shown in Figure 7. From the graph it can be seen that neither one material nor the other can be said to have the best overall thermal-shock resistance. The range of the heat transfer coefficient, h, dictates which material has the highest thermal-shock resistance. This points out the facts that no single parameter can govern the thermal-shock resistance of a material and that the range of heating conditions must be known in order to adequately predict the thermal-shock resistance.

Solution for Variable Material Properties

Since it is known that the physical and mechanical properties of most brittle ceramic materials vary considerably with the temperature, this study was undertaken to examine the effect of this variation of material properties on the prediction of thermal-shock resistance. Aluminum oxide was selected as a representative ceramic, and the mechanical and physical properties were found as functions of the temperature. This variation of the pertinent material properties can be found in Figures 8, 9, 10, 11, 12 and 13. Polynominal and exponential functions were fitted to these curves by the use of a computer program (see Reference 1). The curves resulting from the curve-fitting procedure are shown as dashed lines in Figures 8, 9, 10, 11, 12 and 13. As shown in Figure 10, the curve was fitted to the actual value of α rather than the average $\bar{\alpha}$ since data on the average $\bar{\alpha}$ was not available in the

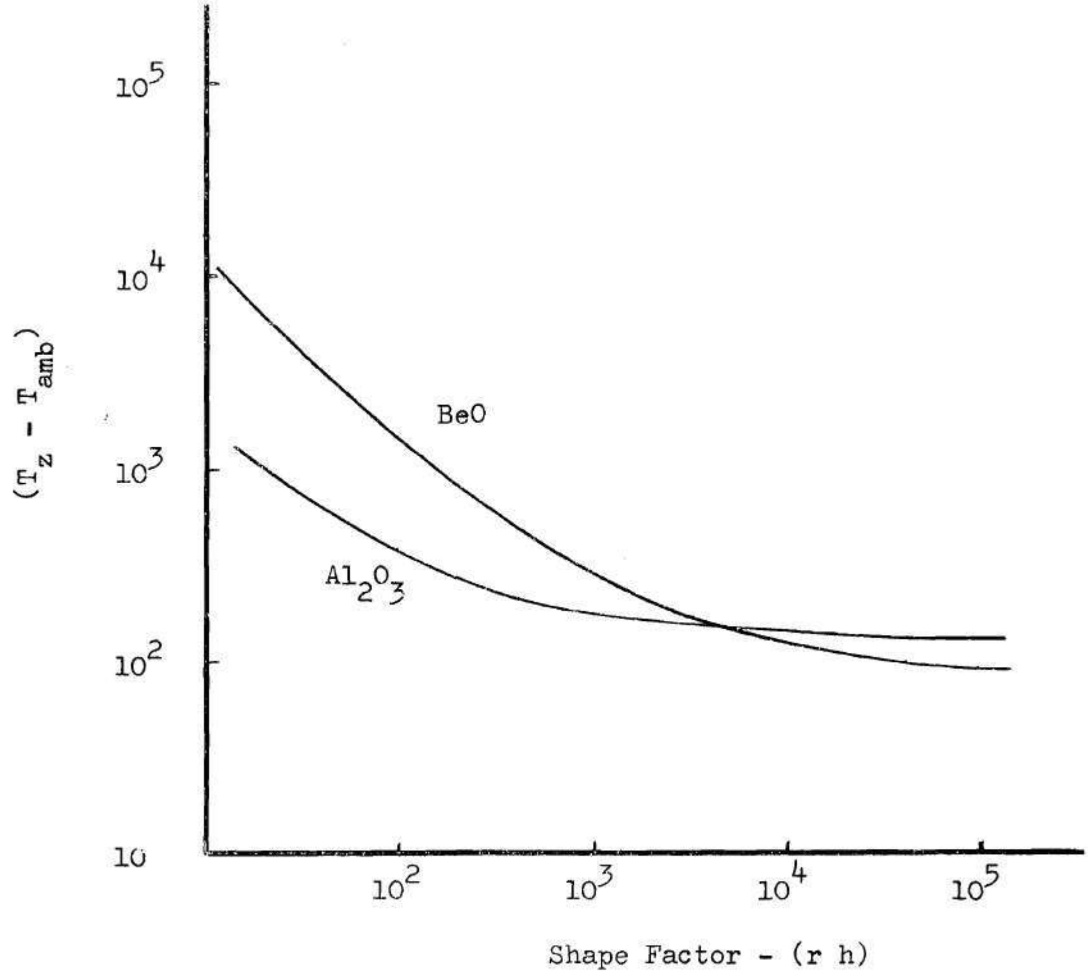

Figure 7. Thermal Shock Resistance versus Shape Factor.

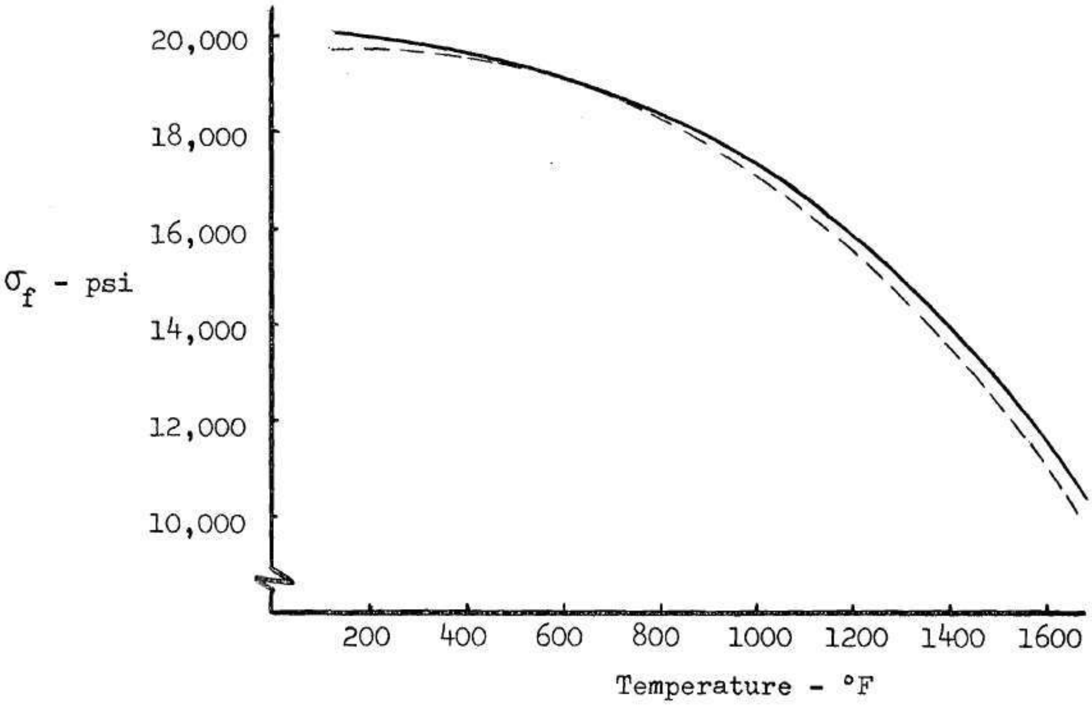

Figure 8. Tensile Fracture Stress versus Temperature - Al_2O_3.

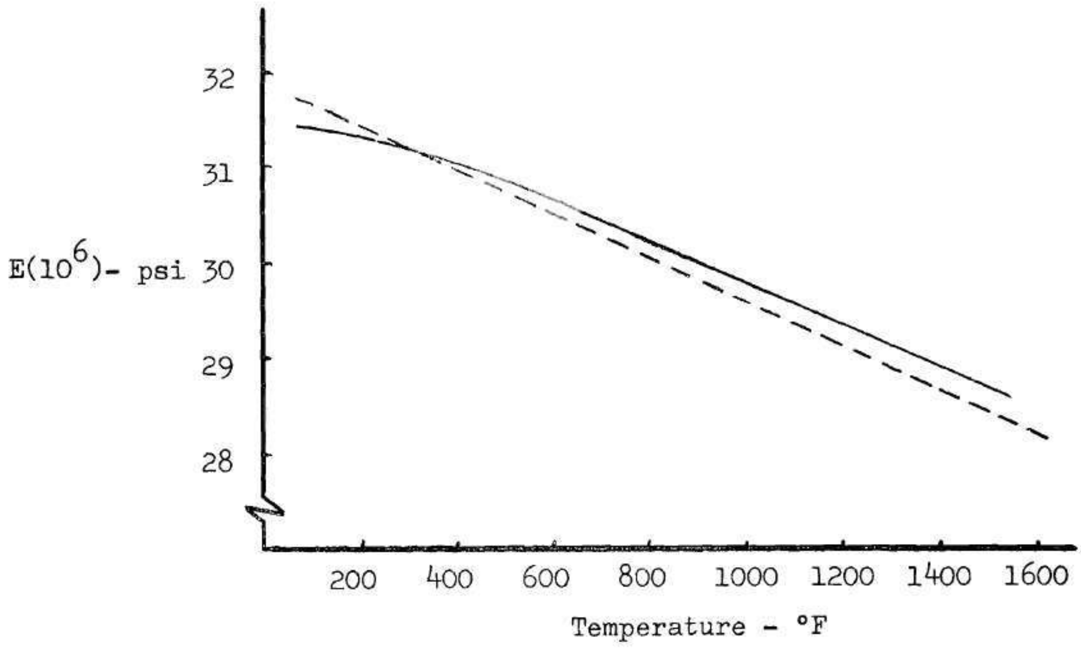

Figure 9. Modulus of Elasticity versus Temperature - Al_2O_3.

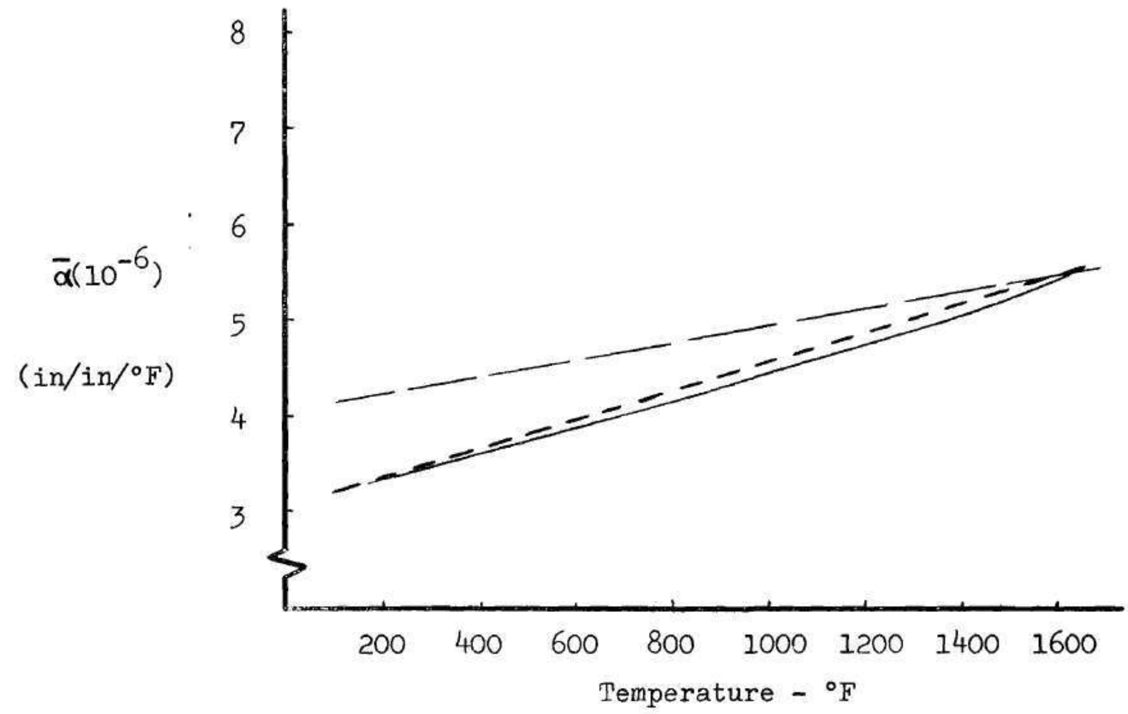

Figure 10. Thermal Coefficient of Expansion versus Temperature - Al_2O_3.

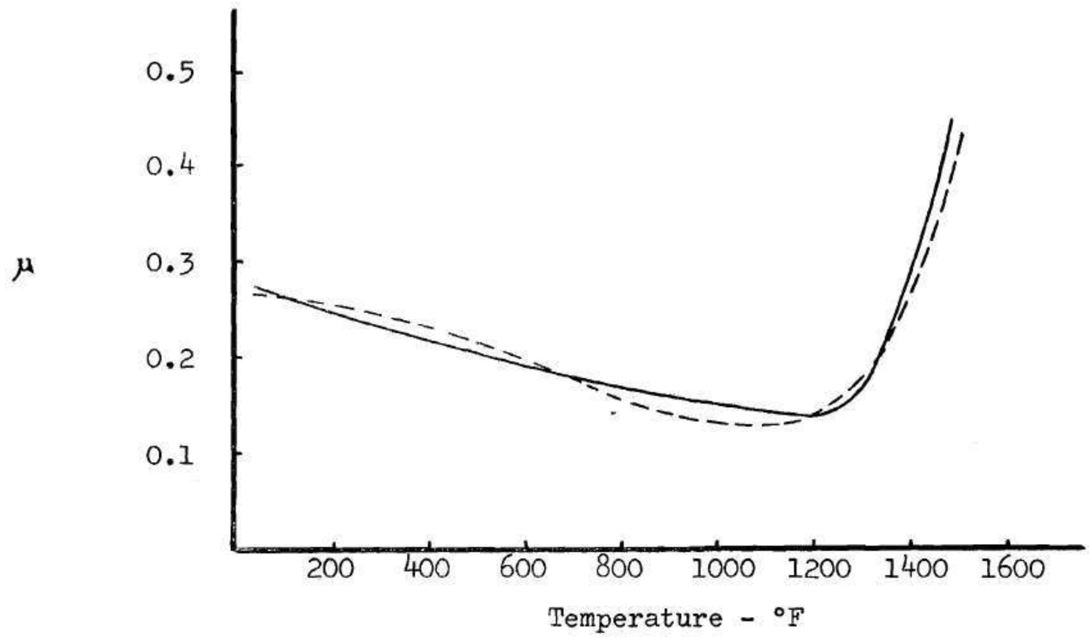

Figure 11. Poisson's Ratio versus Temperature - Al_2O_3.

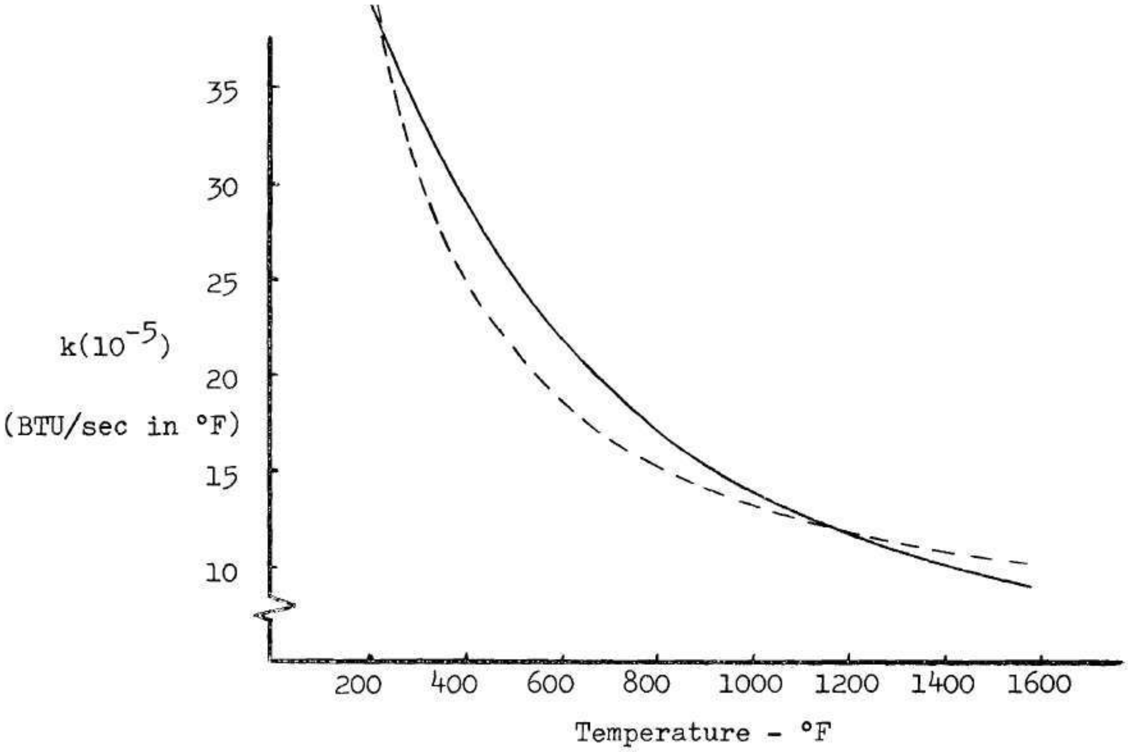

Figure 12. Thermal Conductivity versus Temperature - Al_2O_3.

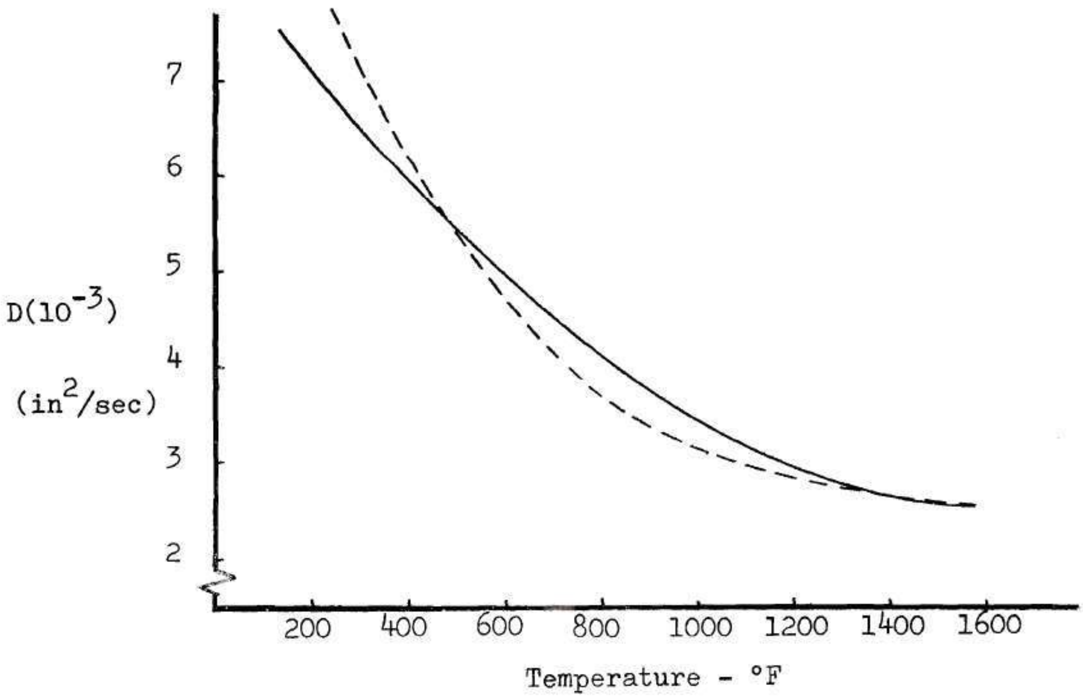

Figure 13. Thermal Diffusivity versus Temperature - Al_2O_3.

literature; however, for the narrow temperature range under consideration (1400° - 1600°F), the resulting error in $\bar{\alpha}$ will be negligable. This fact can be verified in Figure 10 by noting that the curve for $\bar{\alpha}$ (dashed line) is very close to the curve for α in the temperature range under consideration.

As a result of the curve-fitting procedure, the mechanical and physical properties of aluminum oxide were obtained as functions of the temperature. These functions were used in the two computer programs to obtain the "exact" solution for the temperatures and thermal stresses in the disk. For a given initial temperature (T_z = 1600°F), the temperatures and thermal stress were computed for several different values of the heat transfer coefficient, h, and for several different values of the shock temperature, (T_{amb}). From this data, curves of σ^* versus Δt for different values of the heat transfer coefficient were plotted. A set of these curves is shown in Figure 14. The resemblence to the curves shown in Figure 5 can be seen even though the time, Δt, and the heat transfer coefficient, h, are not in non-dimensional form.

Since the temperatures in the disk are known for each value of "h" shown on Figure 14, the fracture stress can be plotted on this same graph. There will be one value of "h" which causes the maximum value of σ^* to be exactly equal to the fracture stress, σ_f^*. For the particular ambient temperature represented in Figure 14 (T_{amb} = 1400°F), the value of the heat transfer coefficient, h, which caused fracture at the instant the maximum stress occurred was 210 BTU/hr ft^2 °F. Using other values of the ambient temperature, T_{amb}, similar plots could be obtained to give the values of the heat transfer coefficient which first caused failure to

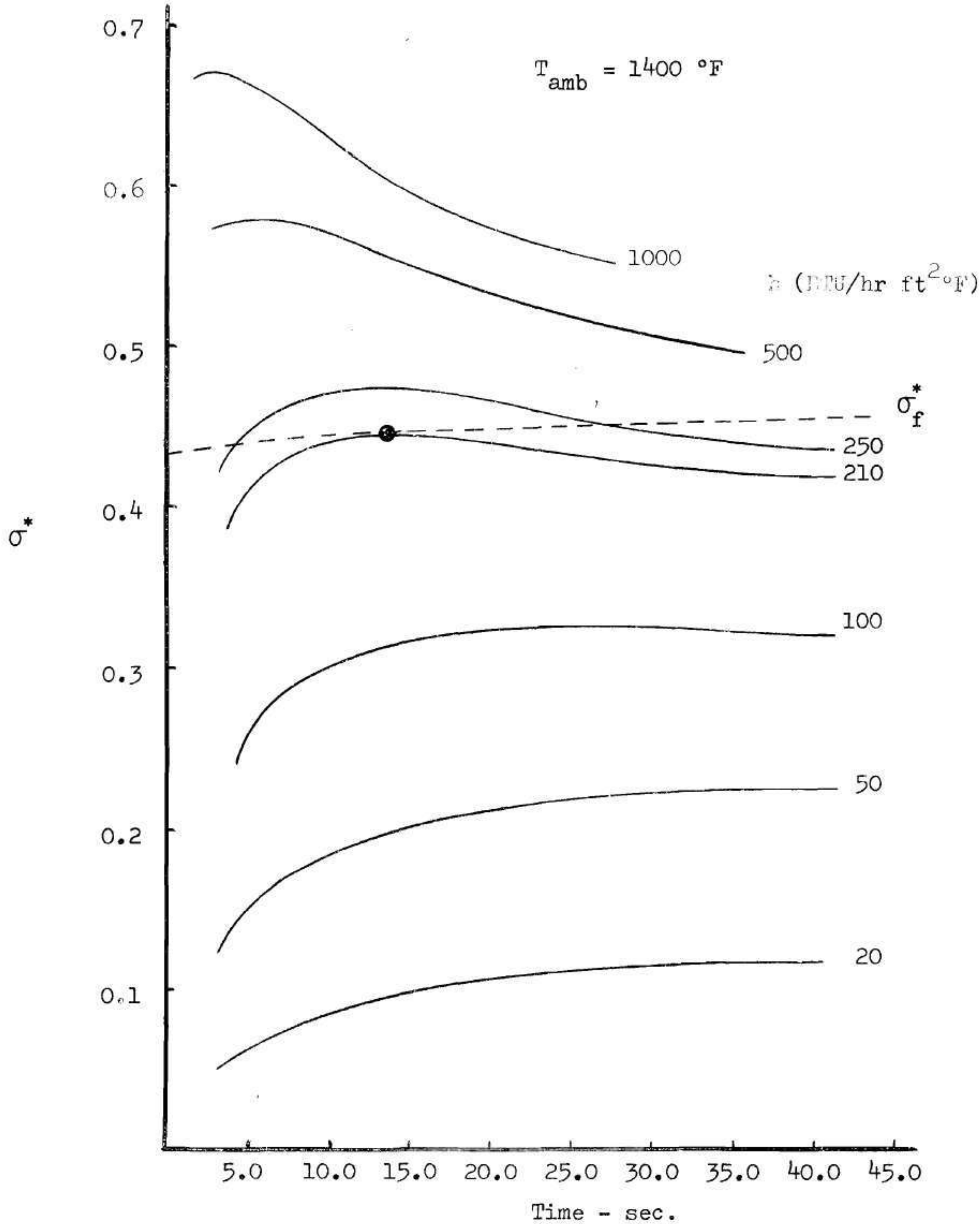

Figure 14. Dimensionless Stress versus Time.

occur. From these results, a plot was made (see Figure 15) of the "thermal shock resistance", $(T_z - T_{amb})$, versus the "shape factor", (r h). This plot enables one to predict the maximum temperature difference an aluminum oxide disk (initially at 1600°F) under an edge quench of a certain severity ("h" given) can be subjected to in order that a tensile fracture occur at the periphery. This plot is usually the final result in a thermal shock analysis.

It is important to note that the thermal shock resistance, as given by the conventional analysis, is merely the temperature difference between <u>any</u> initial temperature, T_z, and the ambient temperature, T_{amb}, because the material properties are independent of the temperature. However, in using the "exact" analysis, the initial temperature must be specified in order to obtain the thermal shock resistance because the mechanical properties are dependent on the initial as well as the subquent temperatures in the body.

Comparison of the Conventional and the Exact Analysis

As presented in this study, the "exact" analysis was rather lengthy; however, once the digital computer programs were set up, it was a simple matter to obtain the necessary data. This indicates that where digital computer equipment is available an "exact" thermal shock analysis of a simple body is quite feasible. However, if hand methods of computation are to be employed, the "conventional" method of assuming constant material properties would be most advantageous.

In order to use the conventional analysis, a temperature at which to evaluate the material properties must be decided upon. The best criterion for selecting this temperature is that it causes the

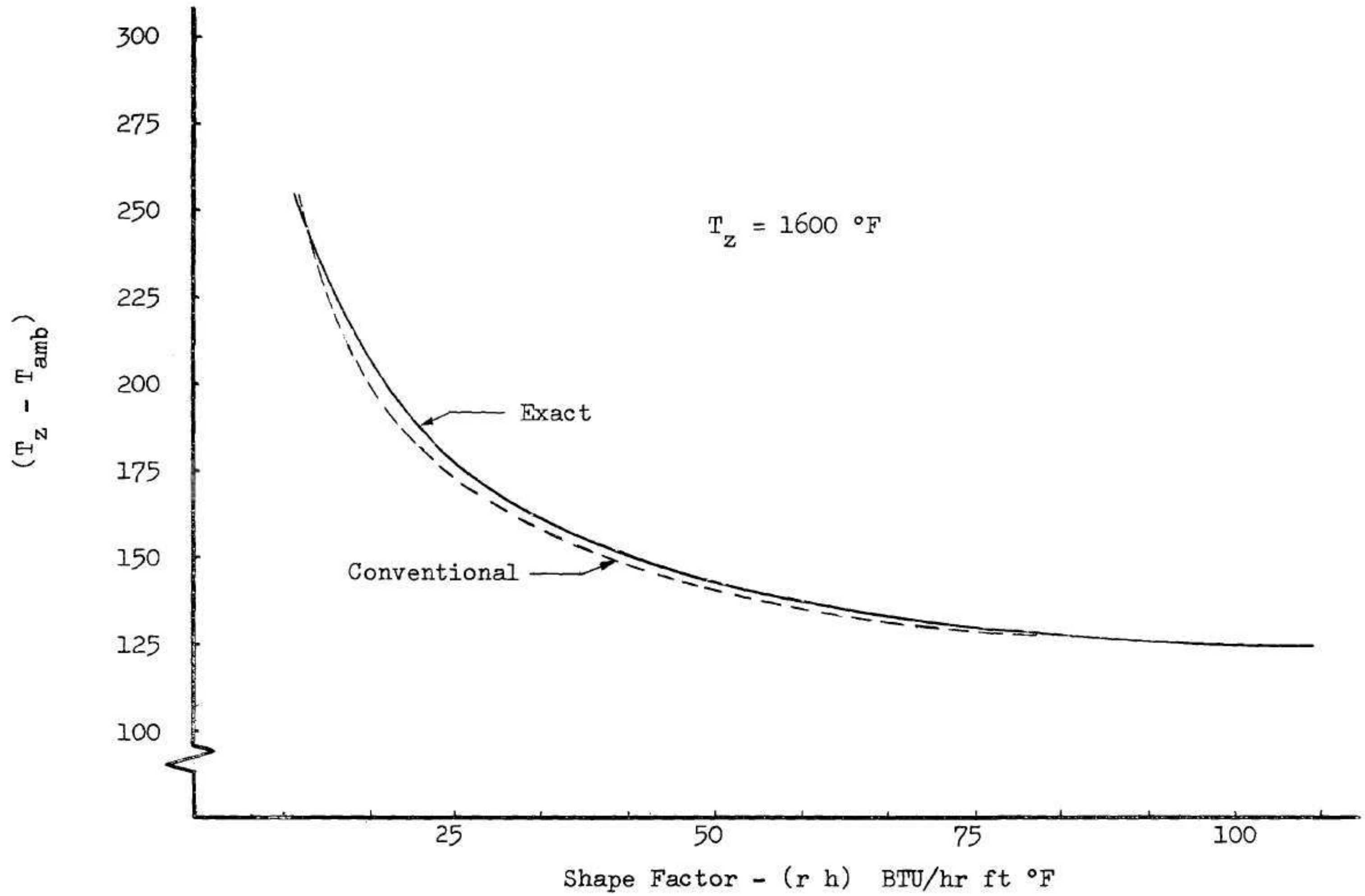

Figure 15. Thermal Shock Resistance versus Shape Factor.

conventional analysis to give the same maximum stress, σ^* , as the exact solution. Just how to find the temperature which best satisfies this criterion is the subject of this section.

Manson[19] suggests that the proper choice for this "mean" temperature would be a temperature midway between the initial temperature, T_z, and the temperature at which the maximum stress occurs, because the maximum stress at the critical point normally occurs before the temperature at this point has changed appreciably. Since this seemed to be the best suggestion made by any of the authors in the current literature, it was decided to use Manson's suggestion. The only drawback in using the mean temperature given by Manson is that an analysis must have been previously performed in order to know the temperatures at which the maximum stress occurs. Of course this presented no problem in the thesis analysis because the temperatures at which the maximum stresses occurred could be found by using the temperatures and stresses from the "exact" analysis; consequently, the "mean" temperatures could be calculated. By evaluating the material properties at these "mean" temperatures, the conventional analysis was made. That is, the temperatures, thermal stresses, and thermal shock resistance were found by solving the differential equations with the material properties held constant. From these solutions, a plot of the "thermal shock resistance",$(T_z - T_{amb})$, versus the shape factor, (r h), was made on Figure 15 to show the agreement between the "exact" and this "conventional" analysis. As can be seen, the results of the conventional analysis are in good agreement with the exact analysis.

As expected, the "mean" temperatures were found to be functions of the shape factor, (r h). If a "mean" temperature factor, K_m, is defined as follows,

$$K_m = \frac{T_m - T_{amb}}{T_z - T_{amb}}$$

then the mean temperature can be easily found once K_m is known. Figure 16 is a plot of the "mean" temperature factor, K_m, versus the shape factor, r h, for T_{amb} = 1400°F. By using this graph, it is possible to find the best "mean" temperature at which to evaluate the temperature dependent properties for any given heat transfer coefficient, h, or shape factor, (r h). However, this plot is restricted to the disk's being at an initial temperature of 1600°F. This means that in order to do a conventional analysis covering a certain temperature range, a family of curves with several different initial temperatures, T_z, as parameters must be obtained.

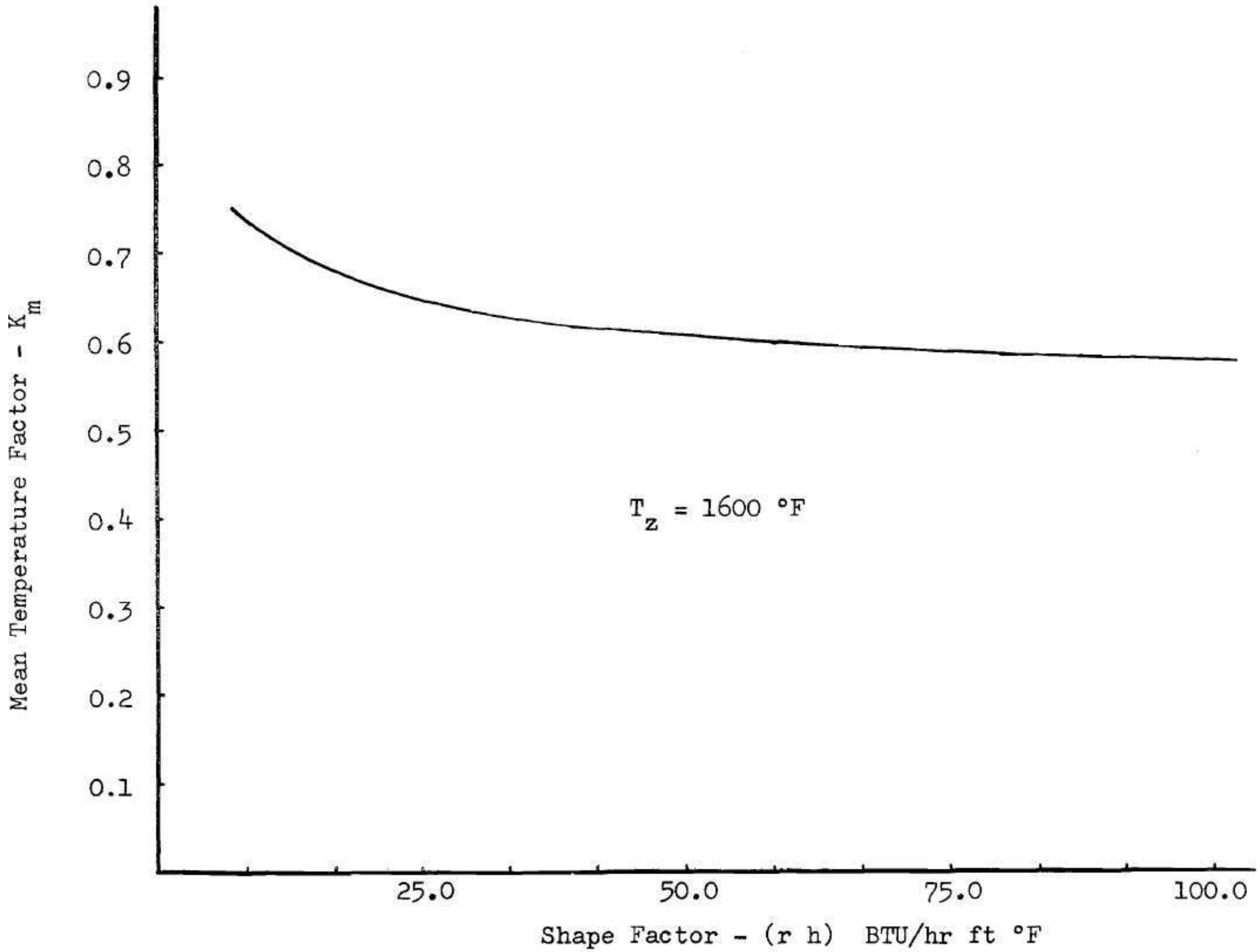

Figure 16. Mean Temperature Factor versus Shape Factor.

CHAPTER VI

DISCUSSION

The conclusions and recommendations that resulted from this study will be presented in this chapter. They will be discussed in two groups - those pertaining to the numerical solution of the differential equations and those pertaining to the thermal shock analysis.

Conclusions

Numerical Solutions

The non-linear heat conduction equation and the plane-stress thermal stress equations in polar coordinates were all successfully solved by numerical techniques. From this analysis, the following statements can be made:

1. It was found that for central finite-difference approximations, the non-linear heat conduction equation in polar coordinates showed signs of being unstable for large values of the heat transfer coefficient (above 5,000 BTU/hr ft^2°F) and large values of the time increment (above 10 seconds). However, stability could be maintained by reducing one of the above factors below its critical value while leaving the other factor in its critical range.

2. At the center of the disk, the instability that resulted from the lower order term in the differential equation was suppressed by finding the limit of this term as $r \rightarrow 0$. This yielded an auxiliary differential equation which was valid only at the center of the disk.

When this differential equation was expressed in finite-difference form, it proved to be stable at $r = 0$

3. The approximation and solution by finite-differences of the equilibrium and compatibility equations for plane thermal stresses proved to be very satisfactory. The solutions of the difference equations checked to within 2 per cent of a trial case where the stresses were known.

Thermal Shock Analysis

In making the investigation into this "exact" analysis of the thermal shock problem, the following conclusions were made:

1. For simple bodies, an "exact" thermal shock analysis could feasibly be made where digital computer equipment is available.

2. The suggestion made by Manson as to the "mean" temperature at which to evaluate the temperature dependent properties in a conventional thermal shock analysis proved to be very satisfactory in the thesis problem where an exact analysis had been previously made. However, in the general conventional analysis, some type of analysis would have to be made before the mean temperatures could be found because the temperatures at the time of the maximum stress are not known. This simply means that a plot of K_m for the particular material must be available or be easily found in order to accomplish a conventional analysis of any accuracy.

3. For more complicated bodies, an exact analysis on a simple body with a similar stress distribution could be made in order to determine the mean temperature coefficient, K_m, as a function of the shape factor, (r h). Then, a conventional analysis on the complicated body could be made from this information.

Recommendations

Numerical Solution

1. Due to the large amount of work already done on the stability of the heat conduction equation in rectangular coordinates, it seems that a stability criterion should be established for this equation in polar coordinates.

2. It is feasible that through the use of an equation like equation (32) and finite-difference approximations which have smaller truncation errors, a completely stable finite-difference approximation to the heat conduction equation in polar coordinates could and should be found.

Thermal Shock Analysis

1. A more comprehensive analysis on the effect of variable properties on thermal shock resistance should be made. That is, several materials should be examined by an "exact" analysis to determine which properties have the most profound effect on thermal shock resistance and what type of variation in the properties causes the largest changes in the thermal shock resistance.

2. An experimental program should be set up to verify the results obtained in the above analysis.

APPENDICES

APPENDIX A

DERIVATION OF THE HEAT CONDUCTION EQUATION

Referring to Figure 17 and assuming no heat transfer in the "Z" direction due to the insulation, the energy balance on the element is

(heat in + heat generated) = (heat out + energy stored)

Using the appropriate symbols, this becomes

$$q_r + q_\Theta + \bar{Q}\ (\text{volume}) = q_{r+dr} + q_{\Theta+d\Theta} + (\text{volume})\ \frac{\partial}{\partial t}\ (\rho c T) \qquad (1A)$$

or

$$q_r + q_\Theta + \bar{Q}\left[(rd\Theta)(dr)(dz)\right] = \left[q_r + \frac{\partial q_r}{\partial r}\ dr\right] + \left[q_\Theta + \frac{\partial q_\Theta}{\partial \Theta}\ d\Theta\right] \qquad (2A)$$

$$+ \left[(rd\Theta)(dr)(dz)\right]\frac{\partial}{\partial t}(\rho c T)$$

Simplifying

$$\bar{Q}\left[(rd\Theta)(dr)(dz)\right] = \frac{\partial q_r}{\partial r}\ dr + \frac{\partial q_\Theta}{\partial \Theta}\ d\Theta + \left[(rd\Theta)(dr)(d\Theta)\right]\frac{\partial}{\partial t}(\rho c T) \qquad (3A)$$

Using the heat conduction law, the following is obtained

$$q_r = -\ k\ A_r\ \frac{\partial T}{\partial r} = -\ k\ (rd\Theta)(dz)\ \frac{\partial T}{\partial r} \qquad (4A)$$

$$q_O = -\ k\ A_\Theta\ \frac{\partial T}{r\partial \Theta} = -\ k\ (dr)(dz)\ \frac{\partial T}{r\partial \Theta} \qquad (5A)$$

Substituting equations (4A) and (5A) into equation (3A), the following

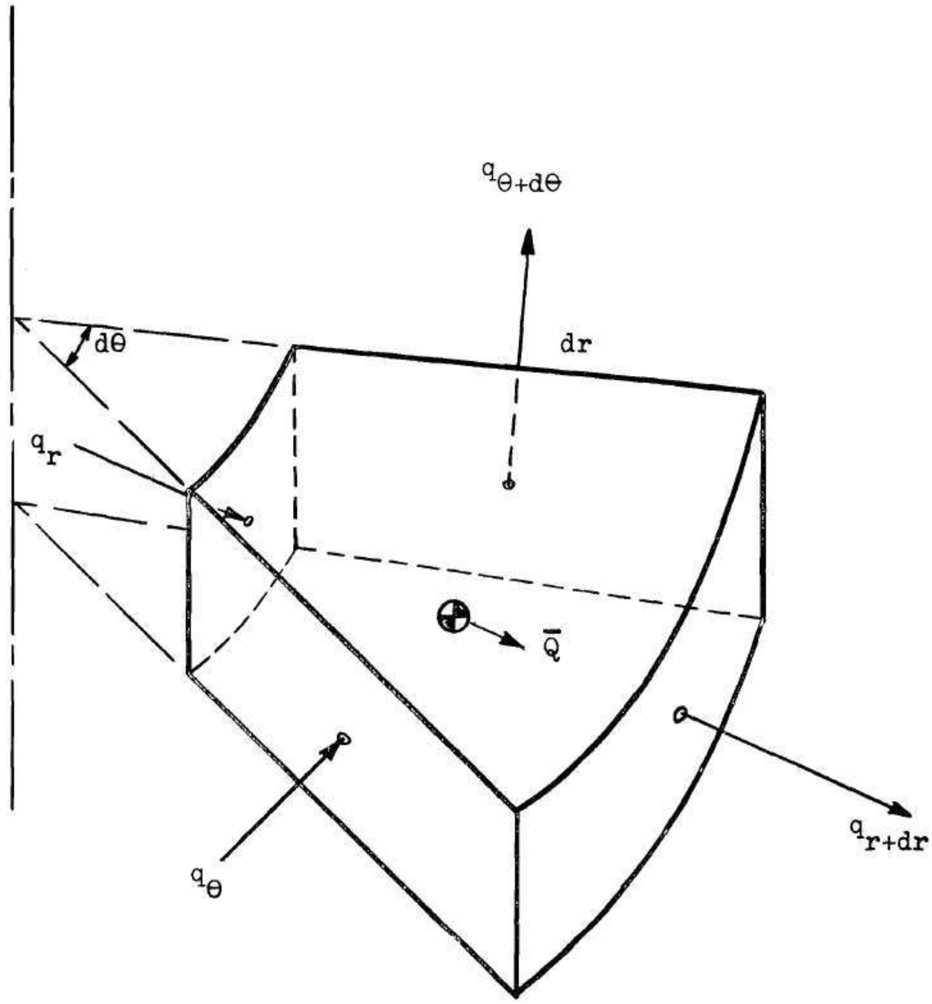

Figure 17. Elemental Particle Showing the Heat Balance.

is obtained

$$\overline{Q}\ \left[(rd\Theta)(dr)(dz)\right] = -\ (d\Theta dz)\ \frac{\partial}{\partial r}\ (k\ r\ \frac{\partial T}{\partial r})\ dr$$

$$-\ (drdz)(1/r)\ \frac{\partial}{\partial \Theta}\ (k\ \frac{\partial T}{\partial \Theta})\ d\Theta + \left[(rd\Theta)(dr)(dz)\right]\ \frac{\partial}{\partial t}(\rho cT) \qquad (6A)$$

Making simplifications and dividing by r, the heat conduction equation becomes

$$\frac{\partial}{\partial r}(k\ \frac{\partial T}{\partial r}) + \frac{k}{r}\ \frac{\partial T}{\partial r} + \frac{1}{r^2}\ \frac{\partial}{\partial \Theta}(k\ \frac{\partial T}{\partial \Theta}) + \overline{Q} = \frac{\partial}{\partial t}(\rho cT) \qquad (7A)$$

APPENDIX B

FINITE-DIFFERENCE FORMULATION OF THE TEMPERATURE PROBLEM

Temperature Equation

Referring to Figure 18, using "central-finite-differences," and writing the finite-difference equations for the general node point defined at,

Radius = $r = (i)$

Time = $t = (j-\frac{1}{2})$

where

(j) = some general point in the disk

$(j-\frac{1}{2})$ = a general point in time which is half way between the $(j)^{th}$ and the $(j-1)^{th}$ points in time.

the heat conduction equation (27) for the thin disk can be written as

$$\frac{T(i+1,j-\frac{1}{2}) - 2T(i,j-\frac{1}{2}) + T(i-1,j-\frac{1}{2})}{(\Delta r)^2} + \frac{1}{r}\left[\frac{T(i+1,j-\frac{1}{2}) - T(i-1,j-\frac{1}{2})}{2\Delta r}\right] \tag{1B}$$

$$= \frac{1}{D}\left[\frac{T(i,j) - T(i,j-1)}{\Delta t}\right]$$

Noting that in general

$$T(m,n-\tfrac{1}{2}) = \frac{T(m,n) + T(m,n-1)}{2} \tag{2B}$$

this type of substitution can be made in equation (1B) to obtain

$$\frac{1}{2}\left[\frac{T(i+1,j) + T(i+1,j-1) - 2\left[T(i,j) + T(i,j-1)\right] + T(i-1,j) + T(i-1,j-1)}{\Delta r^2}\right]$$

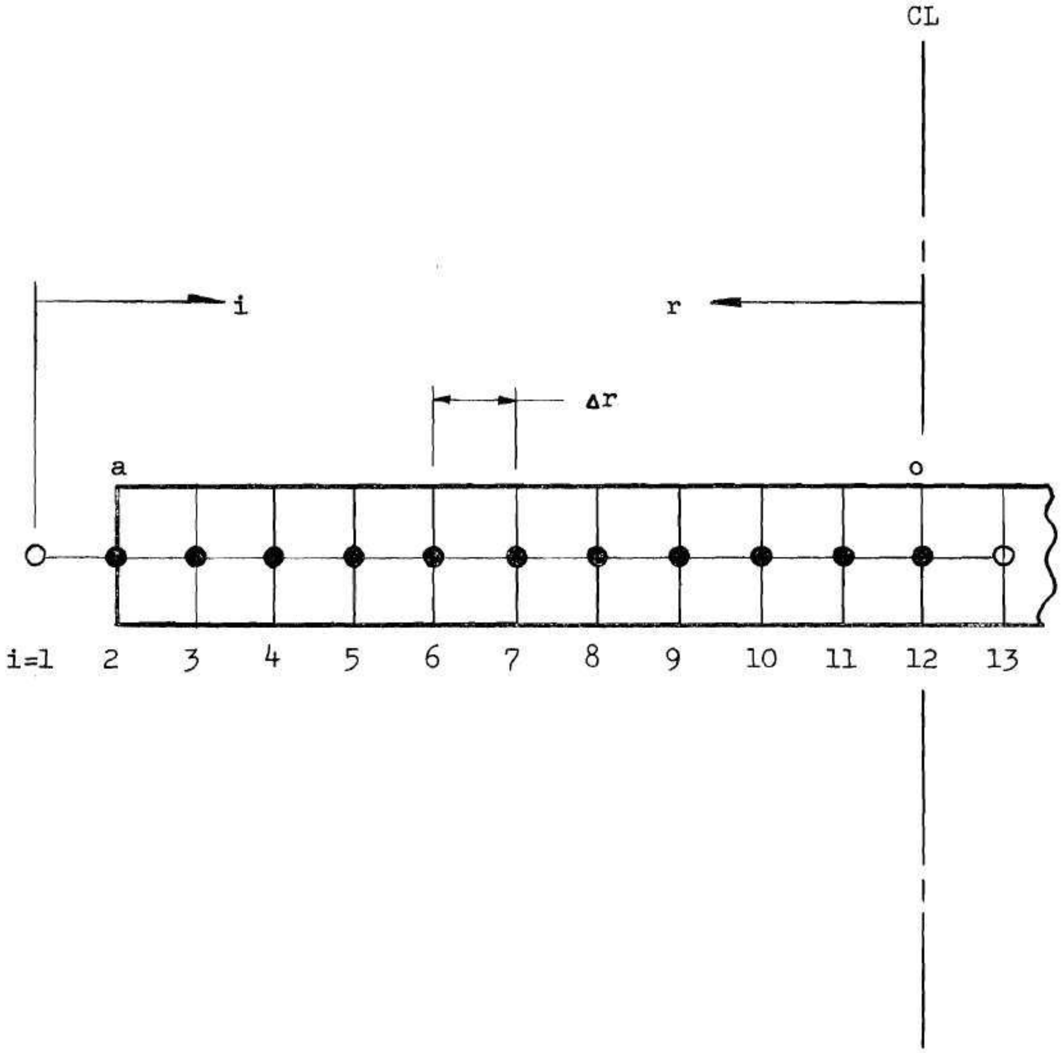

Figure 18. Radial Grid Points in the Disk.

$$+ \frac{1}{2}\left[\frac{1}{r}\left(\frac{\left[T(i+1,j) + T(i+1,j-1)\right] - \left[T(i-1,j) + T(i-1,j-1)\right]}{(\Delta r)}\right)\right]$$

$$= \frac{1}{D}\left[\frac{T(i,j) - T(i,j-1)}{\Delta t}\right] \tag{3B}$$

Solving equation (3B) for T(i,j), the following is obtained

$$T(i,j) = \frac{1}{2 + 2M(i,j)}\left[2\Big(M(i,j)-1\Big)\Big(T(i,j-1)\Big) + \Big(1 + L(i)\Big)\Big(T(i+1,j)\right.$$

$$\left. + T(i+1,j-1)\Big) + \Big(1-L(i)\Big)\Big(T(i-1,j) + T(i-1,j-1)\Big)\right] \tag{4B}$$

where

$$M(i,j) = (\Delta r)^2/(D)(\Delta t)$$

$$L(i,j) = \Delta r/2r \tag{5B}$$

Equation (4B) gives the relation needed to solve for the temperature at all internal nodes or stations in the disk except the center node.

Expressing equation 32 in finite-difference form, the temperature at the center node can be found. Using the same approximations that were used above, the following was obtained.

$$T(12,j) = \frac{2}{M(12,j) + 2}\left[T(11,j) + T(11,j-1) + \frac{T(12,j-1)}{2\left[M(12,j)-2\right]}\right] \tag{6B}$$

Boundary and Initial Conditions

The boundary conditions for the thermal shock problem, equations (28), (29) and (30), can be put in finite-difference form in a manner similar to the temperature equation.

Referring to Figure 18, boundary condition (29) would become

$$\frac{T(13,j) - T(11,j)}{2\Delta r} = 0 \tag{7B}$$

or

$$T(13,j) = T(11,j) \tag{8B}$$

This equation provides a means for determining the value of the temperature at the node point (13,j), which is needed in order to solve for the temperature at the center node point, (12,j).

Also referring to Figure 18, boundary condition (30) would become

$$\frac{T(1,j-\tfrac{1}{2}) - T(3,j-\tfrac{1}{2})}{2\,\Delta r} = \frac{Q}{k} \tag{9B}$$

Using relation (2B), this becomes

$$\frac{1}{2}\left[\frac{T(1,j) + T(1,j-1) - T(3,j) + T(3,j-1)}{2\,\Delta r}\right] = \frac{Q}{k} \tag{10B}$$

or

$$T(1,j) + T(1,j-1) = T(3,j) + T(3,j-1) + \frac{Q}{k}(4\,\Delta r) \tag{11B}$$

Since the heat flux, Q, is determined by the heat transfer coefficient, h, at the surface and the temperature difference between the ambient fluid and the surface of the disk, the following is obtained

$$Q = h\left[TAMB(j-\tfrac{1}{2}) - T(2,j-\tfrac{1}{2})\right] \tag{12B}$$

Again using relation (2B), this becomes

$$Q = h\left[\frac{TAMB(j) + TAMB(j-1)}{2} - \frac{T(2,j) + T(2,j-1)}{2}\right] \tag{13B}$$

Substituting equation (13B) into equation (11B), the following is obtained

$$T(1,j) + T(1,j\text{-}1) = T(3,j) + T(3,j\text{-}1)$$
$$+ \frac{4\,\Delta rh}{k}\left[\frac{TAMB(j) + TAMB(j\text{-}1)}{2} - \frac{T(2,j) + T(2,j\text{-}1)}{2}\right] \qquad (14B)$$

Setting

$$N(i,j) = \frac{\Delta rh}{k} \qquad (15B)$$

Equation (14B) becomes

$$T(1,j) + T(1,j\text{-}1) = T(3,j) + T(3,j\text{-}1)$$
$$+ 4\,N(i,j)\left[\frac{TAMB(j) + TAMB(j\text{-}1)}{2} - \frac{T(2,j) + T(2,j\text{-}1)}{2}\right] \qquad (16B)$$

This equation gives a relation for the temperatures at the imaginary node or station outside the disk, (i,j).

Since equations (4B) and (6B) are used to determine the temperature at all the internal nodes except the node on the periphery of the disk, another equation must be found for this node. Writing this equation for the node on the periphery, the following is obtained:

$$T(2,j) = \frac{1}{2 + 2M(2,j)}\Big[2\,\big(M(i,j)\text{-}1\big)\,\big(T(2,j\text{-}1)\big)$$
$$+ \big(1 + L(2)\big)\,\big(T(3,j) + T(3,j\text{-}1)\big) + \big(1 - L(2)\big)\,\big(T(1,j) + T(1,j\text{-}1)\big)\Big] \qquad (17B)$$

Eliminating the unknown temperatures, T(i,j) and T(i,j-1), by using equation (16B), the following is obtained

$$T(2,j) = \frac{1}{2\left[1 + M(2,j) + N(2,j) - N(2,j)\,L(2)\right]}\Bigg\langle T(2,j-1)\left[2\Big(M(2,j)-1\Big) - \Big(2N(2,j)\Big)\Big(1-L(2)\Big)\right] + 2\Big(T(3,j) + T(3,j-1)\Big) + \Big(2N(2,j)\Big)\Big(1 - L(2)\Big)\Big(TAMB(j) + TAMB(j-1)\Big)\Bigg\rangle \qquad (18B)$$

Therefore, the temperature at all the nodes or stations in the disk can be determined from equations (4B), (6B) and (18B) provided that something is known about the initial temperature. The initial condition is given by equation (28), and this can be simply expressed as

$$T(i,0) = T_z \qquad (19B)$$

where

T_z = the initial constant temperature of the disk at the time increment just prior to the occurrence of the thermal shock.

Temperature at Which Non-Linear Terms Are Evaluated

In order to evaluate the temperature dependent physical and mechanical properties, a temperature at which to evaluate these properties must be decided upon. For very small time increments such as used in the programs ($\Delta t=1.0$), the temperature at any node (i,j) will not change by more than a few degrees from one time increment to the next. For this reason, it was decided to evaluate the temperature dependent terms for the radial node point (i) at the temperature of this same node point at one time increment earlier. That is, the temperature dependent terms for the node point (i,j) are evaluated at the known temperature of the node point (i,j-1).

APPENDIX C

FINITE-DIFFERENCE FORMULATION OF THE THERMAL STRESS PROBLEM

Referring to Figure 19, the terms of equations (34) and (44) could be approximated about point "m" as follows:

$$r_m \approx \tfrac{1}{2}\,(r_i + r_{i-1}) \tag{1C}$$

therefore,

$$(r\,\sigma_r)_m \approx \tfrac{1}{2}\left[(r\,\sigma_r)_i + (r\,\sigma_r)_{i-1}\right] \tag{2C}$$

By using central finite-differences at point "m", a typical derivative term would be

$$\frac{d}{dr}\,(r\,\sigma_r)_m \approx \frac{(r\,\sigma_r)_i - (r\,\sigma_r)_{i-1}}{r_i - r_{i-1}} \tag{3C}$$

The other derivative terms are similarily approximated. The final results of substitutions like expression (2C) and (3C) into equations (34) and (44) are

$$\frac{(r\,\sigma_r)_i - (r\,\sigma_r)_{i-1}}{r_i - r_{i-1}} - \tfrac{1}{2}\left[(\sigma_t)_i + (\sigma_t)_{i-1}\right] = 0 \tag{4C}$$

and

$$\frac{(\sigma_t/E)_i - (\sigma_t/E)_{i-1}}{r_i - r_{i-1}} - \frac{(\mu\,\sigma_r/E)_i - (\mu\,\sigma_r/E)_{i-1}}{r_i - r_{i-1}} + \frac{(\bar{\alpha}\,\Delta T)_i - (\bar{\alpha}\,\Delta T)_{i-1}}{r_i - r_{i-1}}$$

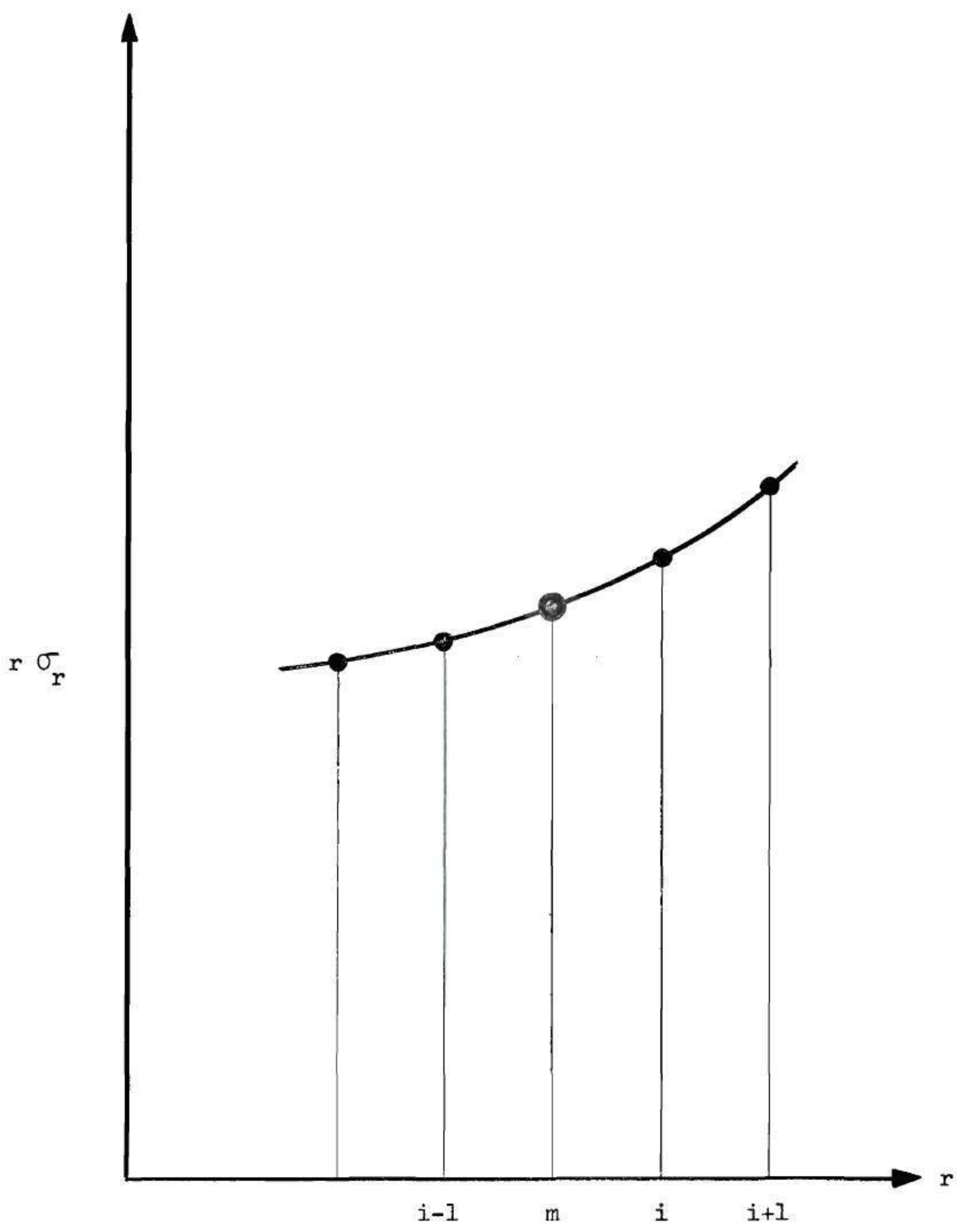

Figure 19. Finite-Difference Grid for the Thermal Stresses.

$$- \frac{1}{2}\left[\frac{(1+\mu)_i(\sigma_r - \sigma_t)_i}{(E\ r)_i} + \frac{(1+\mu)_{i-1}(\sigma_r - \sigma_t)_{i-1}}{(E\ r)_{i-1}}\right] = 0 \qquad (5C)$$

By using central finite-difference approximations for the derivatives at point "m", two equations with four unknowns, namely $(\sigma_r)_i$, $(\sigma_t)_i$, $(\sigma_t)_{i-1}$, and $(\sigma_r)_{i-1}$, have been introduced. By using the boundary conditions (45) and (46), two of the unknowns can be eliminated. If $(\sigma_r)_{i-1}$ and $(\sigma_t)_{i-1}$ are found by using the boundary conditions, then the other unknowns, $(\sigma_r)_i$ and $(\sigma_t)_i$, could be obtained by using equations (4C) and (5C). This is the procedure that will be used.

First, equations (4C) and (5C) will be expressed in a more concise form. Equation (4C) becomes

$$r_i\ (\sigma_r)_i = D_i\ (\sigma_t)_i + \left[r_{i-1}\ (\sigma_r)_{i-1} + D_i\ (\sigma_t)_{i-1}\right] \qquad (6C)$$

Equation (5C) becomes

$$D_i'\ (\sigma_t)_i = C_i'\ (\sigma_r)_i + G_i'\ (\sigma_t)_{i-1} - F_i'\ (\sigma_r)_{i-1} - H_i' \qquad (7C)$$

where

$$C_i' = (\mu/E)_i + \frac{(1+\mu)_i(r_i - r_{i-1})}{2(E\ r)_i} \qquad (8C)$$

$$D_i' = (1/E)_i + \frac{(1+\mu)_i(r_i - r_{i-1})}{2(E\ r)_i} \qquad (9C)$$

$$F_i' = (\mu/E)_{i-1} + \frac{(1+\mu)_{i-1}(r_i - r_{i-1})}{2(E\ r)_{i-1}} \qquad (10C)$$

$$H_i' = (\bar{\alpha}\,\Delta T)_i - (\bar{\alpha}\,\Delta T)_{i-1} \qquad (11C)$$

$$D_i = \frac{1}{2} (r_i - r_{i-1}) \tag{12C}$$

If each term on the right side of equations (6C) and (7C) were known, then σ_t and σ_r at each node or station "i" could be solved by stepping from station to station. By noting that the two equations are linear, the stresses at any station could be expressed in linear terms of the stresses at any other station. Since the solution will start at the center of the disk and proceed outward, the stresses at all stations will be found as a linear function of the stresses at the center station, $(\sigma_t)_o = (\sigma_r)_o$. Therefore, at the general station "i" the stresses are

$$(\sigma_r)_i = (AR)_i \, (\sigma_r)_o + (BR)_i \tag{13C}$$

and

$$(\sigma_t)_i = (AT)_i \, (\sigma_t)_o + (BT)_i \tag{14C}$$

at the station (i-1)

$$(\sigma_r)_{i-1} = (AR)_{i-1} \, (\sigma_r)_o + (BR)_{i-1} \tag{15C}$$

and

$$(\sigma_t)_{i-1} = (AT)_{i-1} \, (\sigma_t)_o + (BT)_{i-1} \tag{16C}$$

where

$(AR)_i$, $(BR)_i$, $(AT)_i$, $(BT)_i$, $(AR)_{i-1}$, etc. are to be determined

At the center of the disk, $\sigma_r = \sigma_t$; therefore, equations (13C) and (14C) could have been expressed in terms of $(\sigma_t)_o$ only.

In order to determine these new constants, equations (13C), (14C), (15C) and (16C) are substituted into the equilibrium equation (6C) and the compatibility equation (7C). After the substitution is made and the terms with and without $(\sigma_t)_o$ are grouped, the following two equations are obtained:

$$\begin{aligned} &\left[r_i(AR)_i - D_i\ (AT)_i - r_{i-1}(AR)_{i-1} - D_i(AT)_{i-1}\right](\sigma_t)_o \\ &+ \ r_i(BR)_i - D_i(BT)_i - r_{i-1}(BR)_{i-1} - D_i(BT)_{i-1} \ = \ 0 \end{aligned} \qquad (17C)$$

and

$$\begin{aligned} &\left[C_i'\ (AR)_i - D_i'(AT)_i - F_i'(AR)_{i-1} + G_i'(AT)_{i-1}\right](\sigma_t)_o \\ &+ \ C_i'(BR)_i - D_i'(BT)_i - F_i'(BR)_{i-1} + G_i'(BT)_{i-1} - H_i' \ = 0 \end{aligned} \qquad (18C)$$

Equation (17C) and (18C) are the new equilibrium and compatibility equations which are in terms of the tangential stress at the center of the disk, $(\sigma_t)_o$. Actually, $(\sigma_t)_o$ is arbitrary as far as these two equations are concerned since it depends entirely on the boundary conditions. That is, $(\sigma_t)_o$ can be made any desired value by merely changing the radial stress at the periphery of the disk. This means that the equilibrium equation (17C) and the compatibility equation (18C) must be independent of the value of $(\sigma_t)_o$; therefore, they must be valid for all $(\sigma_t)_o$. The only way that this could be true is for all four of the coefficient terms in equations (16C) and (17C) to be equal to zero. This gives a set of four equations and four unknowns which can be readily solved for $(AR)_i$, $(BR)_i$, $(AT)_i$ and $(BT)_i$. The solution of these four equations yields the following values of the stress constants

$$(AR)_i = K_i\,(AR)_{i-1} + L_i\,(AT)_{i-1} \tag{19C}$$

$$(AT)_i = K_i'\,(AR)_{i-1} + L_i'\,(AT)_{i-1} \tag{20C}$$

$$(BR)_i = K_i\,(BR)_{i-1} + L_i\,(BT)_{i-1} + M_i \tag{21C}$$

$$(BT)_i = K_i'\,(BR)_{i-1} + L_i'\,(BT)_{i-1} + M_i' \tag{22C}$$

where

$$K_i = \frac{F_i'\,D_i - r_{i-1}\,D_i'}{C_i'\,D_i - r_i\,D_i'} \tag{23C}$$

$$K_i' = \frac{r_i\,F_i' - r_{i-1}\,C_i'}{C_i'\,D_i - r_i\,D_i'} \tag{24C}$$

$$L_i = -\frac{G_i'\,D_i + D_i\,D_i'}{C_i'\,D_i - r_i\,D_i'} \tag{25C}$$

$$L_i' = -\frac{C_i'\,D_i + r_i\,G_i'}{C_i'\,D_i - r_i\,D_i'} \tag{26C}$$

$$M_i = \frac{H_i'\,D_i}{C_i'\,D_i - r_i\,D_i'} \tag{27C}$$

$$M_i' = \frac{r_i - H_i'}{C_i' D_i - r_i D_i'} \tag{28C}$$

Thus, if the four stress constants are known for station (i-1) then they can be found for station (i). Once these stress constants are known at all the stations, the stress distribution can be determined from equation (15C) and (16C).

The stress coefficients for the center station (r = 0) can be found by using the boundary conditions. Writing equation (15C) for the inner station, the following is obtained:

$$(\sigma_r)_o = (AR)_o\,(\sigma_t)_o + (BR)_o \tag{29C}$$

In order for boundary condition (46) to be satisfied, the stress constants must have the values

$$(AR)_o = 1.0 \tag{30C}$$

$$(BR)_o = 0 \tag{31C}$$

In a similar manner, equation (16C) written for the inner radius gives

$$(\sigma_t)_o = (AT)_o\,(\sigma_t)_o + (BT)_o$$

In order for this expression to be consistent, the stress coefficients must have the values

$$(AT)_o = 1.0 \tag{32C}$$

$$(BT)_o = 0 \tag{33C}$$

Now that the stress coefficients for the inner station are known, the coefficients for all the stations are known. There remains but one unknown in the problem, namely $(\sigma_t)_o$. This can be solved for by using the remaining boundary condition (45). Equation (13C) written for the outer station is

$$(\sigma_r)_a = (AR)_a \, (\sigma_t)_o + (BR)_a \tag{34C}$$

therefore, using boundary condition (45)

$$(\sigma_t)_o = - \frac{(BR)_a}{(AR)_a} \tag{35C}$$

Thus, the radial and tangential thermal stresses can be determined from equations (13C) and (14C) for each station since the right hand side of these equations is completely known.

APPENDIX D

Two programs for use on the Burrough's 220 digital computer were written for the numerical solution of the temperature problem and the thermal stress problem. The equations needed in the solution were presented in Chapters III and IV.

The finite-difference grid used for the computer programs was essentially the same grid that was explained in the text of this thesis. Figure 18 shows the grid superimposed on an edge view of the disk. This same grid was used in the temperature and the thermal stress programs.

The programs are presented in the following pages. They should be quite clear to a person familiar with the "ALGOL" computer language because the programs are fully documented with explanations.

```
2COMMENT
2          TRANSIENT TEMPERATURE DISTRIBUTION IN A DISK WITH VARIABLE
2          PROPERTIES CAUSED BY A THERMAL SHOCK ON THE OUTER RIM.
2                                                                        $
2INTEGER   I,J,Q,A,F                                                     $
2ARRAY     TEMP(13,20),RES(13,20),M(13,20),N(13,20),H(13,20),K(13,20),
2          DIFUS(13,20),L(13),TAMB(20)                                   $
2COMMENT
2          THE FOLLOWING ARE THE TIME AND RADIAL INCREMENTS.  (THE UNITS
2          ARE SECONDS AND INCHES)
2                                                                        $
2          DELT=1.0                                                      $
2          DELR=0.1                                                      $
2COMMENT
2          THE FOLLOWING DEFINES THE INITIAL TEMPERATURE...
2                                                                        $
2          TZERO=1600.0                                                  $
2COMMENT
2          SET INITIAL VALUES FOR THE TEMP(I,J) MATRIX...
2                                                                        $
2          FOR I=(1,1,13)                                                $
2          FOR J=(1,1,20)                                                $
2          TEMP(I,J)=TZERO                                               $
2COMMENT
2          THE FOLLOWING DEFINES THE VARIOUS VALUES TO WHICH THE
2          AMBIENT TEMPERATURE IS RAISED...
2                                                                        $
2          F=14                                                          $
2          FOR A=(2,6,F)                                                 $
2BEGIN
2          TFINL=(TZERO-((A)(100.0)))                                    $
2COMMENT
2          BEGIN THE ITERATION...
2                                                                        $
```

```
2           FOR Q=(1,1,15)                                                   $
2BEGIN
2COMMENT
2           THE FOLLOWING ARE THE TEMPERATURE DEPENDENT PROPERTIES...
2                                                                            $
2           FOR I=(1,1,13)                                                   $
2           FOR J=(2,1,20)                                                   $
2BEGIN
2           K(I,J)=((8.1139830**-2)/(TEMP(I,J-1)))+(5.1149491**-5)           $
2           DIFUS(I,J)=((1.9475064)/(TEMP(I,J-1)))+(1.3655940**-3)           $
2           H(I,J)=0.00965
2END                                                                         $
2COMMENT
2           THE FOLLOWING DEFINES THE THREE MODULI OF THE DISK...
2                                                                            $
2           FOR I=(1,1,13)                                                   $
2           FOR J=(2,1,20)                                                   $
2BEGIN
2           M(I,J)=(DELR*2.0)/((DIFUS(I,J))(DELT))                           $
2           N(I,J)=((2.0)(DELR)(H(I,J)))/(K(I,J))
2END                                                                         $
2           L(2)=1.0/20.0                                                    $
2           L(3)=1.0/18.0                                                    $
2           L(4)=1.0/16.0                                                    $
2           L(5)=1.0/14.0                                                    $
2           L(6)=1.0/12.0                                                    $
2           L(7)=1.0/10.0                                                    $
2           L(8)=1.0/8.0                                                     $
2           L(9)=1.0/6.0                                                     $
2           L(10)=1.0/4.0                                                    $
2           L(11)=1.0/2.0                                                    $
2           L(12)=25.0                                                       $
2COMMENT
2           THE FOLLOWING DEFINES THE AMBIENT TEMPERATURE DISTRIBUTION...
```

```
2          TAMB(1)=TZERO                                                     $
2          FOR J=(2,1,20)                                                    $
2          TAMB(J)=TFINL                                                     $
2COMMENT
2          THE FOLLOWING ARE THE BOUNDARY CONDITIONS...
2                                                                            $
2          FOR I=(2,1,12)                                                    $
2          TEMP(I,1)=TZERO                                                   $
2          FOR J=(2,1,20)                                                    $
2BEGIN
2          TEMP(13,J)=TEMP(11,J)                                             $
2COMMENT                                                                     $
2          THE FOLLOWING CALCULATES THE TEMPERATURE AT THE PERIPHERAL
2          NODE...
2                                                                            $
2          TEMP(2,J)=((1.0)/((2.0)(1.0+M(2,J)+N(2,J)-(N(2,J))(L(2)))))((
2          (2.0)(M(2,J)-1.0)-(2.0)(N(2,J))(1.0-L(2)))(TEMP(2,J-1))+(2.0)
2          (TEMP(3,J)+TEMP(3,J-1))+((2.0)(N(2,J))(1.0-L(2)))(TAMB(J)+
2          TAMB(J-1)))
2END                                                                         $
2COMMENT
2          THE FOLLOWING CALCULATES THE TEMPERATURE AT THE CENTER NODE...
2                                                                            $
2          FOR J=(2,1,20)                                                    $
2          TEMP(12,J)=(1.0/((0.5)(M(12,J))+1.0))(TEMP(11,J)+TEMP(11,J-1)
2          +((0.5)(M(12,J))-1.0)(TEMP(12,J-1)))                              $
2COMMENT
2          THE FOLLOWING CALCULATES THE TEMPERATURE AT THE REMAINING
2          NODES...
2                                                                            $
2          FOR J=(2,1,20)                                                    $
2          FOR I=(3,1,11)                                                    $
2          TEMP(I,J)=((1.0)/(2.0+(2.0)(M(I,J))))((2.0)(M(I,J)-1.0)(TEMP(I,
2          J-1))+(1.0+L(I))(TEMP(I+1,J)+TEMP(I+1,J-1))+(1.0-L(I))(TEMP(
```

```
2          I-1,J)+TEMP(I-1,J-1)))                                              $
2END                                                                           $
2COMMENT
2          THE FOLLOWING CALCULATES THE RESIDUALS AT EACH NODE IN ORDER
2          TO DETERMINE IF CONVERGENCE HAS OCCURED...
2                                                                              $
2          FOR J=(2,1,20)                                                      $
2          FOR I=(3,1,12)                                                      $
2          RES(I,J)=((2.0)(M(I,J)-1.0)(TEMP(I,J-1))+(1.0+L(I))(TEMP(I+1,J)
2          +TEMP(I+1,J-1))+(1.0-L(I))(TEMP(I-1,J)+TEMP(I-1,J-1)))
2          -(2.0+(2.0)(M(I,J)))(TEMP(I,J))                                     $
2          WRITE($$TITLE)                                                      $
2          FOR J=(2,1,20)                                                      $
2          FOR I=(2,1,12)                                                      $
2          WRITE($$ANS,FMT)
2END                                                                           $
2OUTPUT    ANS(I,J,TEMP(I,J),RES(I,J),TAMB(J))                                 $
2FORMAT    TITLE(B9,*I*,B4,*J*,B12,*TEMP(I,J)*,B11,*RES(I,J)*,B11,
2          *TAMB(J)*,W0)                                                       $
2FORMAT    FMT(B8,I2,B3,I2,3S20.8,W0)                                          $
2          FINISH                                                              $
```

```
2COMMENT
2         THERMAL STRESSES IN A FLAT DISK DUE TO A THERMAL SHOCK AT
2         EDGE OF THE DISK.   THE TEMPERATURE IN THE DISK MUST BE PUT
2         IN AS DATA.   THE VARIATION OF PROPERTIES WITH TEMPERATURE
2         IS TAKEN INTO ACCOUNT.
2                                                                       $
2INTEGER I,J,A,Z                                                        $
2ARRAY    SIGR(11,10),SIGT(11,10),TEMP(11,10),R(11,10),E(11,10),
2         ALPHA(11,10),NU(11,10),CP(11,10),D(11,10),DP(11,10),
2         FP(11,10),GP(11,10),HP(11,10),K(11,10),KP(11,10),L(11,10),
2         LP(11,10),M(11,10),MP(11,10),AR(11,10),AT(11,10),BR(11,10),
2         BT(11,10)                                                     $
2COMMENT
2         THE FOLLOWING DEFINES THE INITIAL TEMPERATURE AT WHICH THERE
2         IS NO THERMAL STRESS...
2                                                                       $
2         TZERO=1600.0                                                  $
2COMMENT
2         THE FOLLOWING IS THE RADIAL STRESS AT THE OUTER EDGE OF
2         THE DISK...
2                                                                       $
2         SIGRO=0.0                                                     $
2COMMENT
2         THE FOLLOWING DEFINES THE RADIUS AT EACH STATION IN THE DISK...
2                                                                       $
2         FOR J=(1,1,10)                                                $
2BEGIN
2         R(1,J)=0.001                                                  $
2         R(2,J)=0.1                                                    $
2         R(3,J)=0.2                                                    $
2         R(4,J)=0.3                                                    $
2         R(5,J)=0.4                                                    $
2         R(6,J)=0.5                                                    $
2         R(7,J)=0.6                                                    $
```

```
2          R(9,J)=0.8                                                      $
2          R(10,J)=0.9                                                     $
2          R(11,J)=1.0
2END                                                                       $
2COMMENT
2          READ IN THE VALUES OF THE TEMPERATURE...
2                                                                          $
2          READ($$DATA1)                                                   $
2INPUT     DATA1(FOR J=(1,1,10)$FOR I=(1,1,11)$TEMP(I,J))                  $
2COMMENT
2          CALCULATE THE PROPERTIES OF THE DISK WHICH ARE A FUNCTION
2          OF THE TEMPERATURE...
2          R(8,J)=0.7                                                      $
2          FOR I=(1,1,11)                                                  $
2          FOR J=(1,1,10)                                                  $
2BEGIN
2          NU(I,J)=(6.3584796**-10)((TEMP(I,J))*3.0)+(-1.1307450**-6)
2          ((TEMP(I,J))*2.0)+(3.6365165**-4)(TEMP(I,J))+(0.23974927)       $
2          E(I,J)=(-2.3216380**3)(TEMP(I,J))+(3.1899512**7)                $
2          ALPHA(I,J)=(1.4166667**-9)(TEMP(I,J))+(3.0916667**-6)
2END                                                                       $
2COMMENT
2          THE CALCULATION OF THE VARIOUS CONSTANTS USED TO DETERMINE
2          THE STRESSES FOLLOWS...
2                                                                          $
2          FOR I=(2,1,11)                                                  $
2          FOR J=(1,1,10)                                                  $
2BEGIN
2          CP(I,J)=((NU(I,J))/(E(I,J)))+(((1.0)+NU(I,J))(R(I,J)-R(I-1,J))
2          /((2.0)(E(I,J))(R(I,J))))                                       $
2          D(I,J)=(0.5)(R(I,J)-R(I-1,J))                                   $
2          DP(I,J)=(1.0)/(E(I,J))+((((1.0)+NU(I,J))(R(I,J)-R(I-1,J)))
2          /((2.0)(E(I,J))(R(I,J))))                                       $
2          FP(I,J)=(NU(I-1,J))/(E(I-1,J))-((((1.0)+NU(I-1,J))(R(I,J)-
```

```
2           R(I-1,J))))/((2.0)(E(I-1,J))(R(I-1,J))))                  $
2           GP(I,J)=((1.0)/(E(I-1,J)))-((((1.0)+NU(I-1,J))(R(I,J)-R(I-1,J))
2           /(2.0)(E(I-1,J))(R(I-1,J)))                               $
2           HP(I,J)=(ALPHA(I,J))(TEMP(I,J)-TZERO)-(ALPHA(I-1,J))
2           (TEMP(I-1,J)-TZERO)
2END                                                                  $
2           FOR I=(2,1,11)                                            $
2           FOR J=(1,1,10)                                            $
2BEGIN
2           K(I,J)=((FP(I,J))(D(I,J))-(R(I-1,J))(DP(I,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))                               $
2           KP(I,J)=((R(I,J))(FP(I,J))-(CP(I,J))(R(I-1,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))                               $
2           L(I,J)=-((GP(I,J))(D(I,J))+(D(I,J))(DP(I,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))                               $
2           LP(I,J)=-((CP(I,J))(D(I,J))+(R(I,J))(GP(I,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))                               $
2           M(I,J)=((HP(I,J))(D(I,J))+(DP(I,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))                               $
2           MP(I,J)=((CP(I,J))+(R(I,J))(HP(I,J)))/((CP(I,J))
2           (D(I,J))-(R(I,J))(DP(I,J)))
2END                                                                  $
2COMMENT
2           FOR A DISK WITH NO HOLE IN THE CENTER THE FOLLOWING CONSTANTS
2           MUST HAVE THE VALUES SHOWN BELOW...
2                                                                     $
2           FOR J=(1,1,10)                                            $
2BEGIN
2           AR(1,J)=1.0                                               $
2           AT(1,J)=1.0                                               $
2           BR(1,J)=0.0                                               $
2           BT(1,J)=0.0                                               $
2END                                                                  $
2COMMENT
```

```
2          THESE CONSTANTS AT THE OTHER RADIAL STATIONS ARE AS FOLLOWS...
2                                                                          $
2          FOR I=(2,1,11)                                                  $
2          FOR J=(1,1,10)                                                  $
2BEGIN
2          AR(I,J)=(K(I,J))(AR(I-1,J))+(L(I,J))(AT(I-1,J))                 $
2          AT(I,J)=(KP(I,J))(AR(I-1,J))+(LP(I,J))(AT(I-1,J))               $
2          BR(I,J)=(K(I,J))(BR(I-1,J))+(L(I,J))(BT(I-1,J))+M(I,J)          $
2          BT(I,J)=(KP(I,J))(BR(I-1,J))+(LP(I,J))(BT(I-1,J))+MP(I,J)
2END                                                                       $
2COMMENT
2          KNOWING THE RADIAL STRESS AT THE OUTER RIM OF THE DISK (SIGRO)
2          GIVES THE FOLLOWING VALUE OF THE TANGENTIAL STRESS AT
2          THE CENTER...
2                                                                          $
2          FOR J=(1,1,10)                                                  $
2          SIGT(1,J)=((SIGRO)-(BR(11,J)))/(AR(11,J))                       $
2COMMENT
2          SETTING THE RADIAL STRESS AT THE CENTER EQUAL TO THE
2          TANGENTIAL STRESS AT THE CENTER...
2                                                                          $
2          FOR J=(1,1,10)                                                  $
2          SIGR(1,J)=SIGT(1,J)                                             $
2COMMENT
2          CALCULATING THE STRESSES FROM THE KNOWN REFERENCE STRESS AT
2          THE CENTER OF THE DISK - SIGT(1,J)
2                                                                          $
2          FOR J=(1,1,10)                                                  $
2          FOR I=(2,1,11)                                                  $
2BEGIN
2          SIGT(I,J)=(AT(I,J))(SIGT(1,J))+BT(I,J)                          $
2          SIGR(I,J)=(AR(I,J))(SIGT(1,J))+BR(I,J)
2END                                                                       $
2          WRITE($$TITLE)                                                  $
```

```
2COMMENT
2         THE FOLLOWING CALCULATES THE DIMENSIONLESS STRESSES...
2                                                                       $
2         Z=16                                                          $
2         FOR A=(2,2,Z)                                                 $
2BEGIN
2         TFINL=(TZERO)(A)                                              $
2         FOR J=(1,1,10)                                                $
2         FOR I=(1,1,11)                                                $
2BEGIN
2         SIGTB=(SIGT(I,J))/((E(1,1))(ALPHA(1,1))(TZERO-TFINL))         $
2         SIGRB=(SIGR(I,J))/((E(1,1))(ALPHA(1,1))(TZERO-TFINL))         $
2         WRITE($$ANS,FMT)
2END                                                                    $
2END                                                                    $
2OUTPUT   ANS(I,J,R(I,J),SIGT(I,J),SIGR(I,J),SIGTB,SIGRB,TFINL)         $
2FORMAT   TITLE(B2,*I*,B5,*J*,B4,*R(I,J)*,B11,*SIGT(I,J)*,B11,
2         *SIGR(I,J)*,B11,*SIGTB*,B11,*SIGRB*,B11,*TFINL*,W0)           $
2FORMAT   FMT(B1,I2,B4,I2,S10.4,5S20.8,W0)                              $
2         FINISH                                                        $
```

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